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A NEW LOOK AT THE FIFTEEN PUZZLE

EDWARD L. SPITZNAGEL, JR., Northwestern University

1. Introduction. Almost ninety years ago, Sam Loyd invented a very simple puzzle consisting of fifteen blocks to be slid around in a square tray big enough to hold sixteen blocks. For a short time, the puzzle became immensely popular, as people tried to slide the blocks into certain configurations which seemed especially challenging. Then two articles [2, 3] appeared in the American Journal of Mathematics. The first [2] showed that some of those challenging configurations were actually impossible to achieve. The second [3] went on to determine exactly what configurations could be obtained.

As the editors of the Journal pointed out, the puzzle furnished an extremely interesting illustration of the difference between even and odd permutations. Those configurations which could be obtained were closely associated with even permutations of fifteen objects. Those which could not be obtained were closely associated with odd permutations of fifteen objects.

Since that time, interest in the puzzle has diminished, though one can obtain it in several different forms in novelty shops. Also, our view of permutations has changed. We use cyclic notation to write down permutations, and we view the set of all even permutations of n objects as the alternating group A_n, a group with no proper normal subgroups if $n \geqq 5$. It turns out that these new ways of regarding permutations are very closely related to the behavior of the fifteen puzzle, and so in turn the puzzle furnishes a good illustration of these ideas. In this article we examine the puzzle from these new points of view.

2. The puzzle. In its most common form, the puzzle is a shallow square tray about two inches along a side. Within the tray are fifteen blocks each about one half inch square, with two sides grooved and two sides ridged. The blocks are fitted into the tray so that they may be slid freely, but the grooves and ridges prevent their being removed. The blocks are numbered 1, 2, $\cdots$, 15, and it is easy to slide them into numerical order, as in Figure 1. The question then arises: Can one slide the blocks around so as to achieve an arrangement such as in Figure 2, where just the blocks 14 and 15 have been interchanged?

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	

FIG. 1.

1	2	3	4
5	6	7	8
9	10	11	12
13	15	14	

FIG. 2.

3. Configurations unattainable. The very simple argument of this section was given by Ball [1]. Consider the puzzle in any configuration in which the empty space is in the lower right hand corner. Now consider any series of moves rearranging the blocks at the end of which the empty space is returned to the lower right hand corner. The effect of such a sequence of moves is a permutation of the fifteen blocks. We show that the only such permutations which can be obtained by sliding the blocks within the tray are even permutations.

Notice that the permutation of the blocks is completely determined by the path of the empty space. But any path of the empty space can be considered as being composed of several paths, in succession, of an especially simple kind: a closed loop possibly on the end of a path leading from and back to the lower right hand corner. An example appears in Figures 3 to 5, in which the path in Figure 3 can be regarded as the result of taking the path of Figure 4 and then taking the path of Figure 5. The permutation resulting from following one of these simple closed loops is a single cycle, permuting every block along the loop except for the one at the space where the path joining the loop to the lower right hand corner enters. Thus any permutation given by a closed-loop path is cyclic of odd length, so is a product of an even number of transpositions, and so is an even permutation. Since any path can be given as the result of a series of these simple closed loop paths, the corresponding permutation can be written as the product of even permutations, and so must be an even permutation. Thus it is impossible to obtain the configuration in Figure 2 from that of Figure 1, since the permutation, a single transposition, is odd.

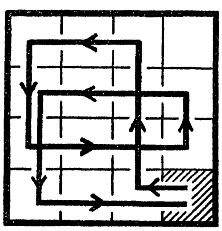

FIG. 3.

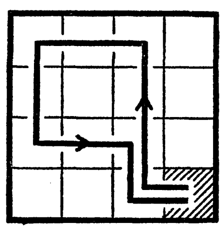

FIG. 4.

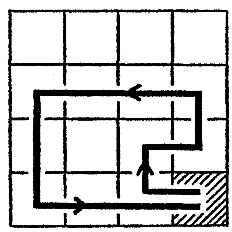

FIG. 5.

4. Configurations attainable. The converse to the preceding result is true, namely if a rearrangement of the blocks which returns the empty space to the lower right hand corner gives an even permutation, then the rearrangement can be achieved by sliding the blocks within the tray. The proof we give here is remarkably closely tied to the fact that the alternating group A_{15} has no proper normal subgroups.

First we show that if three blocks are lined up next to each other in a row or column, then it is possible to permute them cyclically while returning the rest of the puzzle to its initial position. Suppose first that the blocks are lined up in a row. Move the empty space so that it lies either directly above or directly below the rightmost of the three blocks. If it lies below, one cyclic permutation is obtained by the process illustrated in Figures 6 to 9, and the other can be gotten by repeating this process. When this is done, take the empty space back along the route it came, and the rest of the puzzle will return to the initial position. If the empty space lies above, the process to be used is that of Figures 6 to 9 turned upside down. If the blocks are lined up in a column, rotate the puzzle through 90° so that they lie along a row and proceed as above.

Now we show that any three blocks can be permuted cyclically while return-

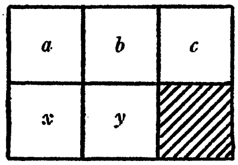

FIG. 6.

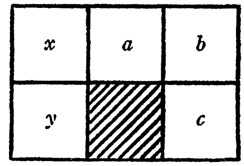

FIG. 7.

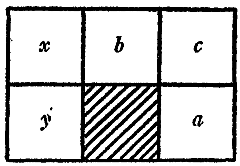

FIG. 8.

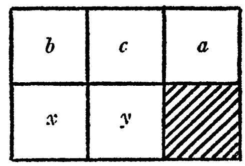

FIG. 9.

ing the rest of the puzzle to its initial position. Let the three blocks to be moved be numbered a, b, c. By 3-cycle permutations in rows and columns, we can move block a to the first row, block b to the second row, and block c to the third row. Then we can move blocks a, b, c over to the column on the extreme right. There they can be permuted cyclically, so that block b replaces a, block c replaces b, and block a replaces c. Then we can reverse the process which brought the blocks to the right hand column, so that block b moves back until it fits into a's initial position, c fits into b's initial position, and a fits into c's initial position. The rest of the blocks have, of course, returned to their initial positions.

Since we can achieve these 3-cycle permutations starting from any initial position, it follows that the set of all configurations which can be obtained from Figure 1 by sliding the blocks and returning the empty space to the lower right hand corner must contain the set of all products of 3-cycles, which is a subgroup of A_{15}. Now every conjugate of the 3-cycle (1, 2, 3) is a 3-cycle, and so this latter subgroup contains the subgroup consisting of all products of conjugates of (1, 2, 3). But this subgroup is normal in A_{15} and so must be all of A_{15}.

Everything which we have done here, with the possible exception of the last two sentences, depends only on results commonly taught in modern algebra courses at the undergraduate level. These two sentences can be replaced as follows. Instead of using the fact that A_{15} has no proper normal subgroups, we prove and use a key lemma from one of the proofs of this fact. We know that the set of configurations attainable from Figure 1 contains the set of all products of 3-cycles. Thus all we have to do is show that every even permutation is a product of 3-cycles. Write out the permutation as a product of an even number of transpositions. It then suffices to show that every product of two transpositions can be written as a product of 3-cycles. If the product consists of two transpositions with a letter in common, we have $(a, b)\ (b, c) = (a, c, b)$. If the product consists of two disjoint transpositions, we have $(a, b)(c, d) = (a, c, b)(b, d, c)$.

Now that we know what rearrangements of Figure 1 we can obtain with the blank space in the lower right hand corner, it is easy to determine whether a given configuration can be achieved, by mentally sliding the blocks so as to put the blank space in the lower right hand corner.

5. Two questions. While the preceding section gives a practical method of proof, that method is not very practical to use if we wish to slide the puzzle into a particular arrangement in a short amount of time. If we define a *move* to mean sliding a block into the blank space next to it, the fewer moves one takes, the more efficient he is at working the puzzle. We can therefore ask: What is the greatest number of moves that will ever be necessary to obtain an attainable configuration from Figure 1? A fairly crude calculation shows that more than 241 moves will never be necessary but the exact number is probably considerably below this.

A second question is: Is it possible to go through the entire list of $\frac{1}{2}\cdot 16!$ configurations attainable without ever repeating one? This would be the most efficient way of obtaining all attainable configurations, since precisely $\frac{1}{2}\cdot 16!$ moves would be used. A check on the time involved shows, however, that if such a thing is possible, it is not very practical. A variant of this second question which should be more vulnerable to analysis is: If we consider only those configurations with the blank space in the lower right hand corner, is it possible to go through the entire list of these without repeating one?

References

1. W. W. R. Ball, Mathematical Recreations and Essays, Macmillan, New York, 1962, pp. 299–300.
2. W. W. Johnson, Notes on the "15" puzzle I, Amer. J. Math., 2 (1879) 397–399.
3. W. E. Story, Notes on the "15" puzzle II, Amer. J. Math., 2 (1879) 399–404.

SPACELAND; AS VIEWED INFORMALLY FROM THE FOURTH DIMENSION

R. A. JACOBSON, Houghton College

Purpose. This note presents a setting for an informal talk at the undergraduate level, hopefully leading to a lively discussion of n-dimensional geometry. For those students who wish to pursue the topic at a more sophisticated level, we suggest treatises by either Coxeter [**2**], including historical notes and a group theory approach, or Sommerville [**3**].

Introduction. In a delightful little book [**1**] by Edwin A. Abbott, we are given an insight into the two-dimensional world of the Flatlander. Introducing a time reference, thus putting Flatland in a third dimension, we make some simple observations that relate certain seemingly different objects. With this in mind we take a retrospective look at our own universe, that of three-dimensional space and time.

Employing the same arguments that we feel should be effective in Flatland, we suggest some intriguing ideas and hope to tempt the reader into further investigation. We conclude with the thought that, fascinating and strange as these observations might be, we suspect that a four-dimensional creature might classify them as "obvious."

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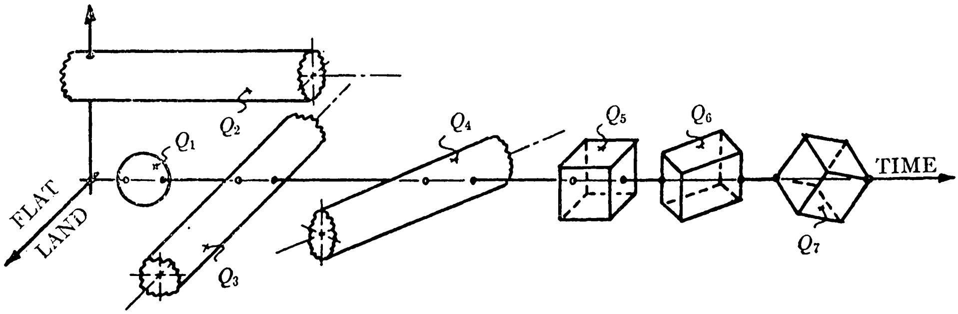

FIG. 1.

Flatland. In establishing a suitable background for our suggestions, we recall some events in our trip to Flatland. During our brief visit, we recorded some observations as this two-dimensional world moved to the right, along a time axis, in a three-dimensional space, Figure 1. In particular, we passed by seven hollow, three-dimensional objects; a sphere, 3 circular cylinders, and 7 unit cubes labeled $Q_1, Q_2, \cdots, Q_7$, respectively. The thing we wish to remember is that although the cylinders and cubes were identical except for orientation, the Flatlanders believed that the seven objects were quite different.

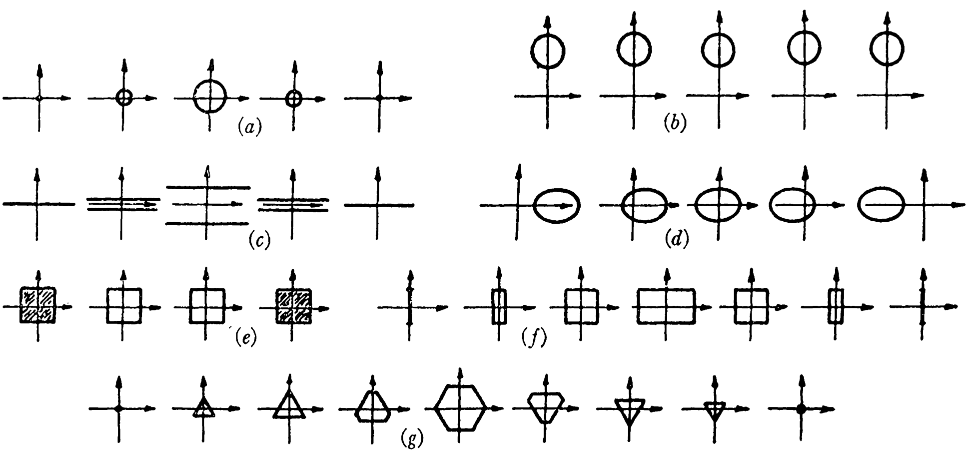

FIG. 2.

Observations. Relying on our familiarity with figures in solid geometry, we shall assume that the Flatlander's observations are evident to the reader. In short, the objects $Q_1, Q_2, \cdots, Q_7$ appeared as the following sequences: concentric circles; a stationary unit circle; two parallel straight lines; a moving ellipse; squares; rectangles; and triangles or hexagons, respectively, as shown in Figure 2.

Since the Flatlanders are not gifted with three-dimensional sight, we find it hard to convince them that observations 2*b*, 2*c*, 2*d* and 2*e*, 2*f*, 2*g* are different views of the same three-dimensional object. In order to establish our assertions, we employ the following analytic argument.

Method of attack. Letting $(x_1\, x_2, \cdots, x_n)$ and $[x_1, x_2, \cdots, x_n]$ denote points and vectors in n-dimensional space, respectively, we attack the general problem as follows:

> Define an object A in n-space, P. Select a time axis through $(0, 0, \cdots, 0)$ and parallel to a given vector. Choose a coordinate system for a moving $n-1$ space, Q, orthogonal to the time axis. In particular, we shall agree that the origin of Q is a function of time and shall lie on the time axis. Finally, we express the intersection of the object A and the Q space in terms of the Q space coordinates and time.

Employing such an analysis, our observations in Flatland can be tabulated as follows:

Sphere: $A_1 = \{(x_1, x_2, x_3) \mid x_1^2 + x_2^2 + x_3^2 = 1\}$

Axes*	Intersection	Description
$t \parallel [0, 0, 1]$ $y_1 \parallel [1, 0, 0]$ $y_2 \parallel [0, 1, 0]$	$Q_1 = \{(0, 0)\}$; $t = -1$ $Q_1 = \{(y_1, y_2) \mid y_1^2 + y_2^2 = 1 - t^2\}$; $t < \lvert 1 \rvert$ $Q_1 = \{(0, 0)\}$; $t = 1$	Point Concentric Circles Point

* "$t \parallel [0, 0, 1]$" is read "t-axis is parallel to vector $[0, 0, 1]$."

Circular Cylinder: $A_2 = \{(x_1, x_2, x_3) \mid x_1^2 + x_2^2 = 1\}$

Axes	Intersection	Description
$t \parallel [0, 0, 1]$ $y_1 \parallel [1, 0, 0]$ $y_2 \parallel [0, 1, 0]$	$Q_2 = \{(y_1, y_2) \mid y_1^2 + y_2^2 = 1\}$	Unit Circle
$t \parallel [0, 1, 0]$ $y_1 \parallel [1, 0, 0]$ $y_2 \parallel [0, 0, 1]$	$Q_3 = \{(y_1, y_2) \mid y_1^2 = 0\}$; $t = -1$ $Q_3 \{(y_1, y_2) \mid y_1^2 = 1 - t^2\}$; $t < \lvert 1 \rvert$ $Q_3 = \{(y_1, y_2) \mid y_1^2 = 0\}$; $t = 1$	Line Two parallel lines Line
$t \parallel [0, \sqrt{2}/2, \sqrt{2}/2]$ $y_1 \parallel [1, 0, 0]$ $y_2 \parallel [0, \sqrt{2}/2, -\sqrt{2}/2]$	$Q_4 = \{(y_1, y_2) \mid 2y_1^2 + (y_2 + t)^2 = 2\}$	Moving ellipse

Hollow Cube: $A_3 = \{(x_1, x_2, x_3) \mid x_i \in \{0, 1\}$ and $x_j \in [0, 1],\ i \neq j\}$

Axes	Intersection	Description
$t \parallel [0, 0, 1]$ $y_1 \parallel [1, 0, 0]$ $y_2 \parallel [0, 1, 0]$	$Q_5 = \{(y_1, y_2) \mid y_i \in [0, 1]\}$; $t = 0$ $Q_5 = \{(y_1, y_2) \mid y_i \in \{0, 1\}$ and $y_j \in [0, 1]\}$; $0 < t < 1$ $Q_5 = \{(y_1, y_2) \mid y_i \in [0, 1]\}$; $t = 1$	Square and interior Square Square and interior

Axes	Intersection	Description
$t \parallel [\sqrt{2}/2, 0, \sqrt{2}/2]$	$Q_6 = \{(y_1, y_2) \mid y_2 \in [0, 1], y_1 = 0\}; t = 0$	Line segment
$y_1 \parallel [\sqrt{2}/2, 0, -\sqrt{2}/2]$	$Q_6 = \{(y_1, y_2) \mid y_2 \in [0, 1], y_1 = \pm t\}; 0 < t \leqq \sqrt{2}/2$	Rectangle
$y_2 \parallel [0, 1, 0]$	$Q_6 = \{(y_1, y_2) \mid y_2 \in [0, 1], y_1 = \pm(\sqrt{2} - t)\}; \sqrt{2}/2 < t < \sqrt{2}$	Rectangle
	$Q_6 = \{(y_1, y_2) \mid y_2 \in [0, 1], y_1 = 0\}; t = \sqrt{2}$	Line segment
$t \parallel [\sqrt{3}/3, \sqrt{3}/3, \sqrt{3}/3]$	$Q_7 = \{(0, 0)\}; t = 0$	Point
$y_1 \parallel [\sqrt{2}/2, -\sqrt{2}/2, 0]$	$Q_7 = \{(y_1, y_2) \mid y_1, y_2$ lie on line segments connecting points $(0, t\sqrt{2})$, $(-t\sqrt{6}/2, -t\sqrt{2}/2$, $(t\sqrt{6}/2, -t\sqrt{2}/2)\}; 0 < t \leqq \sqrt{3}/3$	Equilateral triangle
$y_2 \parallel [\sqrt{6}/6, \sqrt{6}/6, -2\sqrt{6}/6]$	$Q_7 = \{(y_1, y_2) \mid y_1, y_2$ lie on a hexagon, description left to the reader$\}; \sqrt{3}/3 < t < 2\sqrt{3}/3$	Hexagon, regular when $t = \sqrt{3}/2$
	$Q_7 = \{(y_1, y_2) \mid y_1, y_2$ lie on a triangle, description left to the reader$\}; 2\sqrt{3}/3 < t < \sqrt{3}$	Equilateral triangle
	$Q_7 = \{(0, 0)\}, t = \sqrt{3}$	Point

Spaceland. Realizing that the previous results appear strange to the Flatlanders, we are eager to apply the same methods in our own three-dimensional world. Not wishing to spoil the reader's chance of discovery, we shall mention only three rather simple objects in 4-space, indicate a few intersections with 3-space, and note the surprising "equivalence" of seemingly unrelated objects. In particular, we define:

a 4-sphere, $A_4 = \{(x_1, x_2, x_3, x_4) \mid x_1^2 + x_2^2 + x_3^3 + x_4^2 = 1\}$;
a 4-cylinder, $A_5 = \{(x_1, x_2, x_3, x_4) \mid x_1^2 + x_2^2 + x_3^2 = 1\}$; and
a 4-cube, $A_6 = \{(x_1, x_2, x_3, x_4) \mid x_i \in \{0, 1\}$ and $x_j \in [0, 1], i \neq j\}$.

Choosing 3-spaces orthogonal to a particular vector (time axis), we list the corresponding intersection in the following table:

Object	*Vector*	*Intersection*
A_4	any vector	spheres
A_5	[0, 0, 0, 1]	unit sphere
A_5	[0, 0, 1, 0]	cylinders
A_5	[1, 1, 0, 0]	moving ellipsoid
A_6	[0, 0, 0, 1]	solid or hollow cube
A_6	[1, 1, 0, 0]	rectangular parallelopipeds
A_6	[1, 1, 1, 1]	tetrahedrons and octahedrons

We shall let the reader employ our previous analysis to verify that the intersections are correct and that the various objects will appear as pictured in Figure 3. It is somewhat fascinating to propose that figure *3b*, *3c*, *3d* and figure *3e*, *3f*, *3g* are essentially the "same" 4-dimensional objects, viewed from a different reference.

Conclusion. It is hoped that the preceding note might encourage some lively discussion in a number of math clubs. It seems apparent that many, rather interesting, equivalences can be found; the search may be laborious but the results are intriguing.

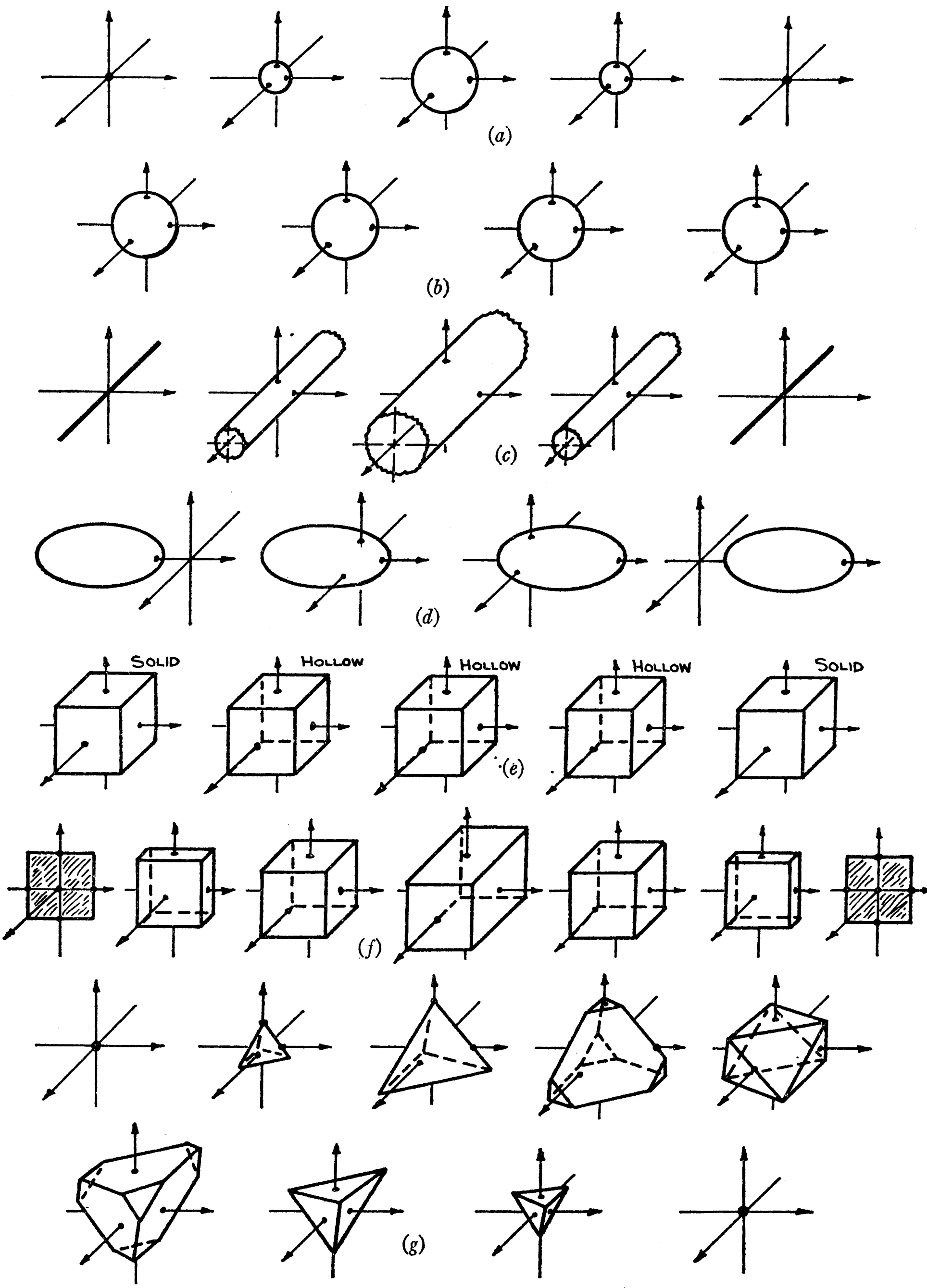

FIG. 3.

References

1. E. A. Abbott, Flatland, a Romance of Many Dimensions, Dover, New York, 1952.
2. H. S. M. Coxeter, Regular Polytopes, Macmillan, New York, 1963.
3. D. M. Y. Sommerville, Geometry of *n* Dimensions, Dover, New York, 1958.

TAKING LIMITS UNDER THE INTEGRAL SIGN

F. CUNNINGHAM, JR., Bryn Mawr College

1. Introduction. This expository paper is addressed to undergraduates for whom "definite integral" means the integral defined by Riemann, and who need occasionally to take limits under the integral sign. This useful manipulation, also called in the context of infinite series "integration term by term," consists in reversing the order of two analytic operations, one being definite integration with respect to one variable, the other taking the limit with respect to another variable. To fix the notation consider one representative case, that of a sequence $f_1, f_2, \cdots$ of functions, each of which is integrable on the same interval $[a, b]$, and such that for each x in $[a, b]$ $\lim_n f_n(x) = f(x)$ exists. We wish to conclude that $\int_a^b f = \lim_n \int_a^b f_n$. It is unfortunately possible for either side of this equation, or both, not to exist, or even for both to exist and be unequal. Examples are given in Advanced Calculus, and a typical one will be found in Section 2 below. This being so, each time this manipulation is used a reason must be given for its correctness in that case. The commonly taught tool for this purpose is the following theorem, which dates from the mid-nineteenth century.

UNIFORM CONVERGENCE THEOREM. *If the sequence $\{f_n\}$ of integrable functions converges uniformly on $[a, b]$ to f, then f is integrable and $\int_a^b f = \lim \int_a^b f_n$.*

Integration theory has value only to the extent that we know some functions which are integrable, i.e., whose integrals exist. Two useful criteria for integrability are that the function be continuous, or that it be piecewise monotone, either of these conditions being sufficient. Half of the above theorem is in effect a third guarantee of integrability. It often happens in applications, however, that the limit function f is known by an explicit formula, is seen to be continuous or piecewise monotone, so that its integrability is known in advance. In that case a much weaker hypothesis than uniform convergence is enough to insure that $\int_a^b f = \lim \int_a^b f_n$. Namely, it is enough to have all the functions bounded by the same constant (see Theorem 1 below), or even less than this (see Theorem 2). Knowing one of these theorems not only obviously increases one's power to take limits in certain cases, and relieves one in other cases of the burden of an irrelevant test for uniform convergence; it also gives one a truer understanding of what makes term-by-term integration give the right answer.

The discovery by C. Arzelà of the bounded convergence theorem, as Theorem 1 is called, preceded by a few years the famous work of Lebesgue at the beginning of this century, which carried the matter farther. By refining the definition of the integral, Lebesgue extended integration to many functions which are too discontinuous to have Riemann integrals, in such a way that the limit function in the bounded (or dominated) convergence theorem is automatically integrable. This made the Riemann integral obsolete. It also eliminated the role of uniform convergence in integration theory, which was only to carry the integrability of the functions in a sequence over to the limit. Unfortunately Lebesgue integration is more complicated than Riemann integration, and this puts Lebesgue's proofs of the theorems in this paper out of reach of the readers to whom this is addressed. References to Arzelà and to later simplifications of his proof by other authors can be found in [1]. The method used here has the virtue of requiring a minimum of previous knowledge. Only the most basic notions of real analysis are used up to Section 4, where the subject takes a slightly more technical turn.

2. The bounded convergence theorem. In order to present the argument in its simplest form, I begin with a theorem which, while good enough for many applications, is not the most general possible. Improvements are discussed in Section 4. As usual, inequalities involving functions asserted to hold on a set A mean that the corresponding inequalities between values of the functions hold for all points in A. When no set A is mentioned, the inequality applies to the whole domain, which is always $[a, b]$. I write $\int$ for $\int_a^b$ and $|f|$ for the function whose value at x is $|f(x)|$.

THEOREM 1. *Let $\{f_n\}$ be a sequence of integrable functions defined on $[a, b]$. Assume*

(i) *For each x in $[a, b]$ $f_n(x) \to f(x)$ (as $n \to \infty$), where f is integrable on $[a, b]$.*
(ii) *There exists a constant K such that $|f_n| \leqq K$ for all n. Then $\int f_n \to \int f$.*

Examples of the following kind help to illuminate the force of hypothesis (ii). Take $[a, b]$ to be $[0, 1]$. Let f be any nonnegative integrable function defined on $[0, 1]$, with $f(0)=0$ and $\int_0^1 f = J \neq 0$; for instance $f(x) = \sin \pi x$. Extend f to $[0, \infty)$ by setting $f(x)=0$ when $x > 1$, and set $f_n(x) = c_n f(nx)$, where $c_1, c_2, \cdots$ are adjustable constants. (The graph of f_n has the same shape as that of f, but compressed into the interval $[0, 1/n]$ and with height changed by the factor c_n.) Regardless how the c_n are chosen, $f_n(x) \to 0$ for each x in $[0, 1]$. The convergence to 0 is uniform if and only if $c_n \to 0$. Hypothesis (ii) is satisfied if and only if the sequence $\{c_n\}$ is bounded. Thus Theorem 1 applies, while the uniform convergence theorem does not, if for instance $c_n = 1$ for all n, and the conclusion of the theorem is easily checked by direct calculation: $\int_0^1 f_n = c_n n^{-1} J \to 0$. By contrast, taking $c_n = n$ in violation of (ii) gives $\int_0^1 f_n = J$. Since $\lim J = J \neq 0$, the conclusion of the theorem is in this case false. (The gap between $\{c_n\}$ bounded and $c_n = n$ is a weakness of the theorem to be remedied later.)

What the theorem is saying is: if you squeeze a tube of toothpaste towards flatness at every point, and if there is a bound on the cross-section of all bulges, then all the toothpaste must come out.

The heart of Theorem 1 is contained in a lemma of Arzelà, which is actually a very special case of the theorem. I shall prove the theorem from the lemma in this section, and in the next section do the interesting part, which is the proof of the lemma. The lemma has to do with subsets of $[a, b]$ of a particularly simple kind, namely finite unions of closed intervals. Call this family of sets $\mathfrak{F}$. For $F \in \mathfrak{F}$ I write $|F|$ to mean the *measure* of F, that is the sum of the lengths of the disjoint closed intervals whose union is F.

LEMMA 1. (Arzelà). *Let $\{F_n\}$ be an infinite sequence of sets belonging to $\mathfrak{F}$. Assume there exists $\epsilon > 0$ such that $|F_n| \geqq \epsilon$ for all n. Then there is some x in $[a, b]$ which belongs to infinitely many F_n.*

To see that this is a special case of Theorem 1, let f_n be, for each n, the characteristic function of F_n, equal to 1 on F_n and 0 everywhere else. Then f_n is integrable; in fact $\int f_n = |F_n|$. To say for some x in $[a, b]$ that $f_n(x) \to 0$ is to say that $x \in F_n$ for only a finite number of n. Thus if we assume the conclusion of the lemma false, then $\{f_n\}$ satisfies (i) of Theorem 1. Since (ii) is trivial for characteristic functions, it would follow from Theorem 1 that $|F_n| \to 0$, contradicting $|F_n| \geqq \epsilon$.

The proof that, conversely, Lemma 1 implies the theorem is as follows: For each n set $g_n = |f_n - f|$. Then $g_n \geqq 0$, $g_n(x) \to 0$ for all x, and $g_n \leqq 2K$. If the theorem were false, then we would have an $\epsilon > 0$ such that $\int g_n \geqq \epsilon$ for infinitely many n. For each of these n apply Lemma 2 below to g_n to obtain a set $F_n \in \mathfrak{F}$ such that $g_n \geqq \epsilon/4(b-a)$ on F_n and $|F_n| \geqq \epsilon/8K$. Then by Arzelà's Lemma some x in $[a, b]$ belongs to infinitely many F_n. For this x we have $g_n(x) \geqq \epsilon/4(b-a)$ infinitely many times, contradicting $g_n(x) \to 0$, and finishing the proof.

LEMMA 2. *Let g be a nonnegative integrable function on $[a, b]$ satisfying $g \leqq K'$ and $\int g \geqq \epsilon$. Then there exists a set F belonging to $\mathfrak{F}$ such that $|F| \geqq \epsilon/4K'$ and $g \geqq \epsilon/4(b-a)$ on F.*

Proof. I shall call a *lower sum* for g any number expressible in the form $\sum_{i=1}^{r} y_i |I_i|$ where $I_1, \cdots, I_r$ are closed intervals, disjoint except for common endpoints, whose union is $[a, b]$ (briefly, a *partition* of $[a, b]$), and $y_1, \cdots, y_r$ are numbers such that $g \geqq y_i$ on I_i for $i = 1, \cdots, r$. By definition $\int g$ is the least upper bound of all lower sums for g. Therefore, since $\int g > \frac{1}{2}\epsilon$, we can find $I_1, \cdots, I_r$ and $y_1, \cdots, y_r$ so that the lower sum $\sum_{i=1}^{r} y_i |I_i| > \frac{1}{2}\epsilon$. For F take the union of those I_i for which the corresponding $y_i \geqq \epsilon/4(b-a)$. Then certainly $g \geqq \epsilon/4(b-a)$ on F. To estimate $|F|$ it is convenient to write $\sum_F$ for sums over only those indices corresponding to intervals included in F, and $\sum_{F'}$ for sums over the remaining indices. Thus $|F| = \sum_F |I_i|$. Now for terms in $\sum_{F'}$, $y_i < \epsilon/4(b-a)$, so

$$\sum_{F'} y_i |I_i| < [\epsilon/4(b-a)] \sum_{F'} |I_i| \leqq \tfrac{1}{4}\epsilon.$$

Therefore, altogether,

$$\tfrac{1}{2}\epsilon < \sum_{i=1}^{r} y_i |I_i| = \sum_{F} y_i |I_i| + \sum_{F'} y_i |I_i| \leqq K' |F| + \tfrac{1}{4}\epsilon$$

from which follows $|F| \geqq \epsilon/4K'$.

3. Arzelà's Lemma. Lemma 2 has an intuitively digestible content. It is a "crowding principle" to the effect that too many large sets cannot fit into a finite space without much overlapping. It can be compared to the elementary combinatorial fact that if $A_1, \cdots, A_m$ are subsets of a finite set S having only N elements, and if each A_i has at least n elements, then some element of S must belong to at least mn/N of the sets A_i. In Arzelà's Lemma the cardinal numbers N and n are replaced by continuous size-measures, $b-a$ and ϵ respectively, and m is infinite.

To show how nontrivial Arzelà's Lemma is, be it remarked that its truth depends on the completeness of the real number system, and therefore so does Theorem 1. I mean this in the strong sense that a context can be envisaged in which these theorems make sense and are false, namely, replace the real interval $[a, b]$ by the rational interval $[a, b]$ consisting of the rational numbers between a and b inclusive. Indeed, for each positive integer n let R_n be the set of all those rational numbers in $[a, b]$ whose fractional representations in lowest terms have denominators $\geqq n$. Then each R_n contains all except a finite number of the rationals in $[a, b]$, and it is easy to construct a set F_n belonging to the rational counterpart of $\mathfrak{F}$, contained in R_n, and satisfying $|F_n| \geqq \frac{1}{2}$. Now any rational number m/k in lowest terms belongs to R_n, and therefore also F_n, only at most for $n=1, 2, \cdots, k$, that is, for a finite number of n. Thus for the rational number system as domain, the sequence $\{F_n\}$ is a counterexample to Lemma 2, and the corresponding sequence of characteristic functions is a counterexample to Theorem 1.

The example just given exhibits the topological ingredient of the theorem. There is also a combinatorial ingredient, which is isolated in the next lemma. The relevant properties of $\mathfrak{F}$ and measure used here are only that $\mathfrak{F}$ is closed under the taking of finite unions and intersections (unfortunately not also under the taking of complements), and two axioms for measure:

1. Monotonicity: If $E_1 \subset E_2$, then $|E_1| \leqq |E_2|$
2. Additivity: $|\phi| = 0$, and for all E_1 and E_2 in $\mathfrak{F}$

$$|E_1 \cup E_2| + |E_1 \cap E_2| = |E_1| + |E_2|$$

If these facts are not considered obvious, they are most easily proved by temporarily enlarging $\mathfrak{F}$ to include all finite unions of intervals (closed or not), so that relative complementation becomes possible in $\mathfrak{F}$. The definition of measure applies equally well to sets in this larger family, and Axiom 2 follows by use of relative complements from its less general form

2′. If $E_1 \cap E_2 = \phi$, then $|E_1 \cup E_2| = |E_1| + |E_2|$

LEMMA 3. *Let $G_0 \in \mathfrak{F}$, and let $\{F_n\}$ be a sequence of sets in $\mathfrak{F}$ contained in G_0 such that $|F_n| \geqq \epsilon$ for all n, where $\epsilon > 0$. Then there exists a nested sequence $G_0 \supset G_1 \supset G_2 \cdots$ of sets in $\mathfrak{F}$ such that $|G_n| \geqq \frac{1}{2}\epsilon$ for all n and every point of $G = \bigcap_{n=1}^{\infty} G_n$ belongs to infinitely many F_n.*

Proof. Choose a sequence $\delta_1, \delta_2, \cdots$ of numbers such that $\epsilon > \delta_1 > \delta_2 > \cdots > \frac{1}{2}\epsilon$; for instance $\delta_n = (\frac{1}{2} + \frac{1}{4}n^{-1})\epsilon$ would do. Now $|F_1 \cup \cdots \cup F_n|$ increases with n and is bounded above by $|G_0|$ (Axiom 1), so converges to some limit L_1. Let $G_1 = F_1 \cup \cdots \cup F_{n_1}$, where n_1 is chosen so large that $|G_1| > L_1 - (\epsilon - \delta_1)$. Then for all $n > n_1$ we have $|G_1 \cup F_n| \leqq |F_1 \cup \cdots \cup F_n| \leqq L_1 < G_1 + \epsilon - \delta_1$. Hence (Axiom 2)

$$|G_1 \cap F_n| = |G_1| + |F_n| - |G_1 \cup F_n| \geqq |G_1| + \epsilon - |G_1| + \epsilon - \delta_1) = \delta_1.$$

We can now repeat the construction, using the sequence of sets $G_1 \cap F_{n_1+1}$, $G_1 \cap F_{n_1+2}, \cdots$, and with ϵ replaced by δ_1. Namely $|G_1 \cap (F_{n_1+1} \cup \cdots \cup F_{n_1+n})|$

converges to some limit L_2. We define $G_2=G_1\cap(F_{n_1+1}\cup\cdots\cup F_{n_2})$ taking n_2 so large that $|G_2|>L_2-(\delta_1-\delta_2)$. Then we conclude as before that for $n>n_2$ $|G_2\cap F_n|\geqq\delta_2$. In general, when we have found G_k and n_k, the construction gives an integer $n_{k+1}>n_k$ and a set $G_{k+1}=G_k\cap(F_{n_k+1}\cup\cdots\cup F_{n_{k+1}})$ belonging to $\mathfrak{F}$, contained in G_k, and such that $|G_{k+1}\cap F_n|\geqq\delta_{k+1}$ for all $n>n_{k+1}$. Now we check the other properties required in the lemma. Clearly $|G_k|\geqq\delta_k>\frac{1}{2}\epsilon$. Also, since $G_k\subset F_{n_{k-1}}+1\cup\cdots\cup F_{n_k}$, every point in G_k belongs to F_n for some n in the range $n_{k-1}<n\leqq n_k$. If $x\in G=\bigcap_{k=1}^{\infty}G_k$, this is true for every k, and therefore x belongs to F_n for infinitely many different n. Thus Lemma 3 is proved.

To finish the proof of Arzelà's Lemma, we apply Lemma 3 with $[a, b]$ for G_0; we then only need to make sure that G is not empty. But this, the topological ingredient, amounts to the compactness of the interval $[a, b]$, G being the intersection of a nested sequence of nonempty (see their measures) closed sets. For completeness I include a quick proof of the famous theorem used here. Let $x_n=\inf G_n$. It exists and belongs to G_n because G_n is nonempty, bounded, and closed. The sequence $\{x_n\}$ is increasing, because the sequence $\{G_n\}$ is nested, and therefore, being bounded above by b, converges to a limit x. For any n and all $k>n$ we have $x_k\in G_k\subset G_n$; whence $x\in G_n$, because G_n is closed. Since this applies to all n, $x\in G$.

4. The dominated convergence theorem. Both hypotheses of Theorem 1 are more restrictive than necessary, but the statement of a more general theorem involves additional technicalities. The aim of this section is to evolve Theorem 2 below by making the necessary small changes in the proof of Theorem 1.

To start with hypothesis (i), it can be relaxed so as to permit a set, provided it is not too large, of exceptions to the rule of pointwise convergence. A set S is said to be *of measure zero* if for every $\epsilon>0$ there exist open intervals $I_1, I_2, \cdots$ (finitely or infinitely many) covering S such that $\sum_{n=1}^{\infty}|I_n|<\epsilon$. For example, all finite sets and all denumerably infinite sets are of measure zero. Because of the Heine-Borel Theorem, any closed bounded set of measure zero can be covered by a finite number of open intervals $I_1, \cdots, I_N$ with $\sum_{n=1}^{N}|I_n|<\epsilon$.

The phrase "almost everywhere" is commonly used as an abbreviation for "with the exception of a set of measure zero."

As the first step in proving Theorem 2 let us show that Theorem 1 remains valid when hypothesis (i) is replaced by

(i′) $f_n(x)\to f(x)$ almost everywhere in $[a, b]$, f being integrable.

The proof of Theorem 1 clearly applies equally to this new version if we obtain in Arzelà's Lemma the stronger conclusion:

The set of x in $[a, b]$ belonging to infinitely many F_n is not of measure zero.

To prove this, use Lemma 3 as before, and then show that G is not only not empty, but not even of measure zero. (It was from foresight of this strengthening that the residual $\frac{1}{2}\epsilon$ was saved in Lemma 2.) Indeed, suppose G is of measure zero. Since G is closed and bounded we can find a finite number of open intervals $I_1, \cdots, I_N$ covering G with $\sum_{n=1}^{N}|I_n|<\frac{1}{4}\epsilon$. The complement of $I_1\cup\cdots\cup I_N$ relative to $[a, b]$ is a set H belonging to $\mathfrak{F}$, and $|H|>b-a-\frac{1}{4}\epsilon$. Now for all k,

$|G_k \cap H| = |G_k| + |H| - |G_k \cup H| > \frac{1}{2}\epsilon + b - a - \frac{1}{4}\epsilon - (b-a) = \frac{1}{4}\epsilon$. Thus $\{G_k \cap H\}$ is a nested sequence of nonempty closed sets, so $G \cap H = \bigcap_{k=1} G_k \cap H$ is not empty. But this contradicts the fact that G is covered by $I_1, \cdots, I_N$ and hence G is not of measure zero.

Turning now to hypothesis (ii) of Theorem 1, we shall see that it can be relaxed so as to permit in place of the constant bound K for all the f_n a (possibly unbounded) dominating function k, provided k is not too large. As a measure of the size of k, I shall use its lower integral, defined in terms of lower sums in the same way as the integral, but without requiring that the integral exist. (The lower integral is the Riemann integral only if it has the same value as the upper integral defined similarly by approximation from above. A function which is unbounded above has no upper sums and hence no upper integral.) For any nonnegative function g, the class of lower sums of g is not empty since it contains 0. If this set is bounded above we define the lower integral $\underline{\int} g$ of g on $[a, b]$ to be its least upper bound. Otherwise we write $\underline{\int} g = \infty$.

THEOREM 2. *Let f_n be a sequence of integrable functions on $[a, b]$. Assume*
(i′) *$f_n(x) \to f(x)$ almost everywhere in $[a, b]$, f being integrable.*
(ii′) *There exists a nonnegative function k such that $\underline{\int} k < \infty$ and $|f_n| \leqq k$ for all n. Then $\int f_n \to \int f$.*

When applying this theorem it is up to the user to produce a suitable function k. If a *bounded* k exists which dominates all the f_n then the extra generality of (ii′) over (ii) is not needed, because k can be replaced by its constant upper bound. To exploit the generality of Theorem 2 one needs a supply of unbounded functions with finite lower integrals. Fortunately there are many. Suppose, for instance, that $\lim_{x \to a+} k(x) = +\infty$, that k is integrable on $[a+\epsilon, b]$ for every $\epsilon > 0$, and that $\int_a^b k$ is a convergent improper integral. Then it is easy to show that $\underline{\int} k \leqq \int_a^b k < \infty$. (In fact the two integrals are equal.)

Example. Returning to the example in Section 2, take $c_n = n^p$ for an exponent $p > 0$. Then $\{f_n\}$ is not uniformly bounded, and Theorem 1 does not apply. But if K is an upper bound for f on $[0, 1]$ then, taking into account that $f_n(x) = 0$ for $x \geqq 1/n$, it is clear that $f_n(x) \leqq Kx^{-p} = k(x)$. Now $\underline{\int}_0^1 k = \int_0^1 Kx^{-p} dx < \infty$ if and only if $p < 1$, and this is precisely the condition under which $\int_0^1 f_n \to 0$, as is verified by direct calculation.

The proof of Theorem 2 imitates that of Theorem 1, but requires an extra step. After defining $g_n = |f_n - f|$ as before and, assuming the conclusion false, taking ϵ appropriately, one cannot immediately apply Lemma 2, because the functions g_n are perhaps not bounded. Note however that f, being integrable, is bounded so that by adding a constant if necessary we can assume $g_n \leqq k$. The strategy is to cut the tops off the g_n to make a uniformly bounded sequence, being careful not to cut off so much as to lose the force of $\int g_n \geqq \epsilon$ (for infinitely many n). The next lemma shows this is possible. I write $u \wedge v$ to mean the smaller, and $u \vee v$ to mean the larger of two numbers u and v. (If $u = v$, then $u \wedge v = u \vee v = u = v$.) Similarly $f \wedge g$ means the function whose value at x is always $f(x) \wedge g(x)$. Thus for a constant K, the truncated function $g \wedge K$ agrees with g wherever $g(x) \leqq K$ and has the value K everywhere else. The nonnegative function $g-g\wedge K$ represents the top of g which has been cut off at the level K.

LEMMA 4. *Let k be a nonnegative function with $\underline{\int} k<\infty$, and let $\epsilon>0$. Then there exists a constant K so large that for any nonnegative integrable function $g \leqq k$ $\int(g-g\wedge K)<\epsilon$.*

Proof. Let $\sum_{i=1}^{r} y_i|I_i|$ be a lower sum for k such that $\sum_{i=1}^{r} y_i|I_i| > \underline{\int} k-\epsilon$, and let K be the largest of the numbers $y_1, \cdots, y_r$ appearing in it. To show that this K works, let g be integrable, $0 \leqq g \leqq k$. If any lower sum of $g-g\wedge K$ is added to $\sum_{i=1}^{r} y_i|I_i|$, the resulting number is still a lower sum for k. This is because if $\Delta y_i \leqq g(x)-g(x)\wedge K$, then $y_i+\Delta y_i \leqq K+g(x)-g(x)\wedge K=g(x)\vee K \leqq k(x)$. Thus $\underline{\int} k-\sum_{i=1}^{r} y_i|I_i|$ is an upper bound for the lower sums of $g-g\wedge K$, and so is as big as their least upper bound, namely $\int(g-g\wedge K)$. Since $\underline{\int} k-\sum_{i=1}^{r} y_i|I_i|<\epsilon$, this proves the lemma.

To finish the proof of Theorem 2, apply Lemma 4 with $\frac{1}{2}\epsilon$ in place of ϵ to obtain K such that for all n $\int(g_n-g_n\wedge K)<\frac{1}{2}\epsilon$ and, hence, $\int(g_n\wedge K)>\int g_n-\frac{1}{2}\epsilon \geqq \frac{1}{2}\epsilon$. Now Theorem 1, in its strengthened form, applies to the uniformly bounded sequence $\{g_n\wedge K\}$ to give a set, not of measure zero, of points x in $[a, b]$ where $g_n(x)\wedge K$ does not converge to 0. For these x it is clear that also $g_n(x)$ does not converge to 0, so the theorem is proved.

5. Applications. I shall indicate only briefly some consequences of Theorems 1 and 2.

THEOREM 3. *Let $f(x, t)$ be a bounded function which is continuous in t for each x and integrable in x for each t. Then $F(t)=\int_a^b f(x, t)dx$ is continuous.*

Proof. To prove continuity at t_0 apply Theorem 1 to $f_n(x)=f(x, t_n)$ where $\{t_n\}$ is an arbitrary sequence converging to t_0.

THEOREM 4. *Let $f(x, t)$ be integrable in x for each t and have a bounded partial derivative $f_t(x, t)$ which is integrable in x for each t. Then $F'(t)=d/dt \int_a^b f(x, t)dx =\int_a^b f_t(x, t)dx$.*

Proof. Set $f_n(x)=n[f(x, t_0+n^{-1})-f(x, t_0)]$. Then (i) $f_n(x)\to f_t(x, t_0)$ for each x, and (ii) $|f_n(x)| \leqq |f_t(x, t_0+\theta n^{-1})| \leqq K$, by the mean value theorem and the boundedness of f_t. Apply Theorem 1.

THEOREM 5. *Let $f(x, y)$ be bounded, and suppose the iterated integrals $\int_c^d[\int_a^b f(x, y)dx]dy$ and $\int_a^b[\int_c^d f(x, y)dy]dx$ both exist. Then they are equal.*

Proof. By existence of the first integral, for instance, I mean that $f(x, y)$ is integrable in x for each y, and that the resulting function $F(y)=\int_a^b f(x, y)dx$ is integrable. Let $I_1, \cdots, I_r$ be a partition of $[a, b]$ and let $x_1\in I_1, \cdots, x_r\in I_r$. Then $S(Y)=\sum_{i=1}^{r} f(x_i, y)|I_i|$ is an approximation to $F(y)$ for each y, while $\int_c^d S(y)dy=\sum_{i=1}^{r}\int_c^d f(x_i, y)dy|I_i|$ is an approximation to the second iterated integral in the theorem. Repeat this for a sequence of partitions with mesh max $|I_i|$ tending to 0 to form a sequence of functions $S_n(y)$ converging for each

y to $F(y)$, and apply Theorem 1. For hypothesis (ii) in Theorem 1 we have $|S_n(y)| \leqq K(b-a)$ where K is a bound for $|f(x, y)|$.

THEOREM 6. *Let* $\{f_n\}$ *be a sequence of functions defined on* $[a, \infty)$ *and integrable on* $[a, b]$ *for all* $b>a$. *Assume*

(i'') $f_n(x) \to f(x)$ *almost everywhere in* $[a, \infty)$, f *being integrable on every finite interval.*

(ii'') *There exists* k *defined on* $[a, \infty)$ *such that* $\int_a^\infty k$ *is convergent and* $|f_n| \leqq k$ *for all* n. *Then* $\int_a^\infty f_n \to \int_a^\infty f$.

Proof. These integrals are absolutely convergent by (ii''). Given $\epsilon>0$ find X so large that $\int_X^\infty k < \frac{1}{4}\epsilon$. Then apply Theorem 2 on the interval $[a, X]$ to find N so large that $\left|\int_a^X f_n - \int_a^X f\right| < \frac{1}{2}\epsilon$ when $n>N$. Then for these n we have

$$\left|\int_a^\infty f_n - \int_a^\infty f\right| \leqq \left|\int_a^X f_n - \int_a^X f\right| + \int_X^\infty |f_n| + \int_X^\infty |f| < \epsilon.$$

As a final example I shall use Theorem 6 to evaluate the conditionally convergent integral $\int_0^\infty x^{-1} \sin x dx$ which turns up in the theory of Fourier series and integrals. A formal trick which works is:

$$\begin{aligned}\int_0^\infty \frac{\sin x}{x} dx &= \int_0^\infty \sin x \int_0^\infty e^{-xt} dt dx \\ &= \int_0^\infty \left[\int_0^\infty \sin x e^{-xt} dx\right] dt \\ &= \int_0^\infty (1+t^2)^{-1} dt = \tfrac{1}{2}\pi.\end{aligned}$$

The change in the order of integration can be explained as follows. Let

$$\begin{aligned}f_n(t) &= \int_0^n \sin x e^{-xt} dx \\ &= (1+t^2)^{-1}[1 - (t \sin n + \cos n)e^{-nt}].\end{aligned}$$

Evidently $|f_n(t)| \leqq (1+t^2)^{-1}[1+(t+1)e^{-t}] = k(t)$, and $\int_0^\infty k$ converges. Thus Theorem 6 gives

$$\begin{aligned}\int_0^\infty \int_0^\infty \sin x e^{-xt} dx dt &= \lim \int_0^\infty f_n \\ &= \int_0^\infty (1+t^2)^{-1} dt\end{aligned}$$

as required. Note that $f_n(0) = 1 - \cos n$ does not converge, so that the "almost everywhere" version of hypothesis (i) is used. The convergence of $f_n(t)$ to $(1+t^2)^{-1}$ is not uniform even on $(0, \infty)$.

Reference

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MATHEMATICAL THEORY OF THINK-A-DOT

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1. Introduction. *Think-a-Dot* is an inexpensive mathematical toy recently put on the market. Its manufacturer claims (with considerable justice, in this author's opinion) that it illustrates many principles of digital computers. Its design also raises a number of interesting mathematical questions in its own right. Some of those are discussed in this article.

The results given here are new, since *Think-a-Dot* is new. The writer believes that they are neither trivial nor obvious. Yet the mathematical foundation required to obtain them is so minimal that this paper might have been written by a high school student! All that is required is to know that the sum of two integers is even if and only if either both summands are even, or both are odd. It is the author's hope that because of this very bare set of prerequisites, this paper may prove understandable to readers down to the junior high school level. To this end, we have tried to avoid unnecessary advanced mathematical terminology. We thereby illustrate that mathematical research can go on at almost any level.

2. Description of Think-a-Dot. *Think-a-Dot* is a box, at the top of which are three holes into any one of which a marble may be inserted. Figure 1 gives a general view of the device. When a marble is inserted, it moves under gravity power down through a set of inclined planes, and emerges on a track at the bottom. Where two planes meet internally there is a "switch" which directs the marble to either the right or left hand emerging path from the switch. There is also a switch at each entry port at the top. Each time a marble goes through a switch, it reverses the switch setting, so that the next marble will go the other way. This is the only way switches may be changed after an initial pattern is set up. On a vertical face of the toy, the states of all switches are displayed through color-coded openings each of which shows either a yellow or blue dot. As marbles are successively dropped through, the color pattern of the dots changes.

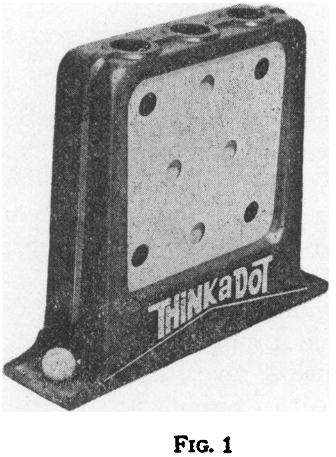

FIG. 1

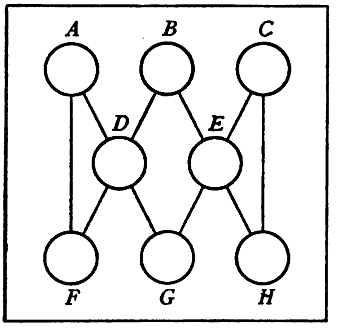

FIG. 2

3. Example. Figure 2 shows the vertical face of *Think-a-Dot* with the eight switches and the available paths between them. Suppose all switches were originally set to the left-hand branches. If a marble were dropped into the center hole, it would pass through switch B and emerge at the left, continue on down to switch D on the inclined plane schematically shown; again emerge on the left, and finally go on to F, where it would at last come out onto the output track from the left hand side of F. In following this path, it would cause the three switches traversed, B, D, and F to reverse their settings. If, then, a second marble were dropped into the center hole, it would now emerge on the right of B (since that switch had been reversed by the first marble) and go to switch E. This switch would still be set for a left hand branch, so the marble would leave it going to the left and pass through G, and come out onto the track. After these two operations, switch B would have been reversed twice, and hence would be in its original state. Switches D, E, F, and G would each have been actuated once, and hence be reversed. And switches A, C, and H would not have been changed at all. The new dot pattern would differ from the original in positions D, E, F, and G.

4. First problem. The manufacturer suggests that an initial dot pattern be set up by tilting the device to one side or the other. The user can then select an arbitrary new pattern, and try to attain it by successive marble drops. (Hence the name.)

The question immediately arises, is every pattern obtainable from every other? Surprisingly, the answer to this is "no," as we shall now show. This means that a user following the manufacturer's instructions can very well find himself engaged in a task that is impossible of fulfillment!

In proving this, it is first convenient to define a new term. We shall say two patterns of switch settings are in *near-agreement* if they agree on all switch positions except the lower center, switch G. This is not the same thing as saying that they agree except for (any) one switch. The term "near-agreement" of two patterns means that they differ specifically on G and on no other switch.

We can now prove:

THEOREM 1. *If any initial pattern is given, then the pattern in near-agreement with it cannot be obtained by marble drops.*

Proof. We shall use the so-called indirect method in which we assume temporarily that the result can be achieved and then show that this leads to a contradiction.

Suppose, therefore, that a pattern is given and there is a sequence of marble drops that converts it to the pattern in near-agreement. Consider first switch A. This is actuated only by marble drops into the left hand hole in the top. Since this switch must be returned to its original state at the end of the operation, the number of marbles dropped into hole A must be even, say $2k$ where k is an integer to be found (if possible). Because A reverses setting at each passage of a marble, we can see that half the time, the exiting marble from A will go to D, and half the time to F. Hence D and F will each be actuated just k times as a result of drops into the left hand hole. These latter two switches can also be actuated by other marbles, a point we shall take care of shortly.

A similar argument shows that the number of drops in the center must be even, say $2m$; and also the number of drops on the right, say $2n$.

Returning to D, we observe that its total number of actuations from either A or B will be $(k+m)$. But again, this must be even, since D is returned to its original state at the end. Let us say

(1) $$k + m = 2s.$$

Similarly, analyzing E, we conclude that

(2) $$m + n = 2t.$$

Looking now at the lower three switches, we have at F just k inputs from A, and s from D.

Since F is not to be changed,

(3) $$(k + s) \text{ is even.}$$

Arguing similarly at H, we find

(4) $$(n + t) \text{ is even.}$$

And finally since G *is* to change, we observe that

(5) $$(s + t) \text{ is odd.}$$

Now there are two possibilities. Either k is even or it is odd.

(6) First suppose k is even.

(7) Then from (1) and (6), m is even

(8) From (3) and (6), s is even

(9) From (2) and (7), n is even

(10) From (4) and (9), t is even

But from (8) and (10), $(s+t)$ is even, contradicting (5). Hence k could not be even.

The reader can quickly verify in like manner that the initial assumption that k is odd leads to the conclusion that s and t are both odd, which again violates (5). Hence the required numbers k, m, n cannot be found, and the transformation desired is impossible. This proves the theorem.

5. Mathematical structure of the patterns. Faced with this disappointing result, we now naturally turn to such questions as the following: Under what conditions can a given pattern be obtained from another? When the transformation is possible, find an algorithm (that is, a systematic procedure) to achieve it.

In order to answer problems of this sort, we now build up a somewhat more elaborate mathematical machinery of the dot patterns.

LEMMA 1. *Any desired pattern of the three switches A, D, F can be obtained from any initial pattern, with between* 0 *and* 7 *drops in the left hand hole (and no drops in the other holes).*

Proof. We shall show that as marbles are dropped in the left hand hole, the three switches progress in sequence through the eight different patterns they can take. This being true, it does not matter where we start in showing it. Suppose then that the switches are initially all set to the left. On succeeding passes, it can be verified directly that the patterns assumed will be as tabulated below.

Switch \ Iteration	0	1	2	3	4	5	6	7	8
A	L	R	L	R	L	R	L	R	L
D	L	L	R	R	L	L	R	R	L
F	L	R	L	R	R	L	R	L	L

For example, on iteration 3, the marble will enter A, which is set to the right. It will leave A on the right hand plane, leading to D. At D, it will be directed to the right since D is shown set to the right on this iteration. It will therefore go on to G, and drop onto the track from there. In this path, it goes through A, D, and G which are therefore reversed at the next iteration, while the other switches are unchanged. Since the table includes only A, D, and F, we find, at iteration 4, that A and D have changed, and F has not.

By this laborious but straightforward process, it can be shown that the table is correct. On examination of it, we can see that the original pattern reappears on iteration 8, and at intermediate steps, all other possible patterns of the designated three switches occur. This completes the proof.

LEMMA 2. *Any desired pattern of the three switches C, E, H, can be obtained from any initial pattern, with between* 0 *and* 7 *drops in the right hand hole.*

This is a symmetrical statement and the proof is similar to Lemma 1.

THEOREM 2. *Any pattern (of all eight switches) can be brought into agreement or near-agreement with any other pattern with* 15 *marble drops or less.*

Proof. First, with zero or one drop in the center hole, obtain conformity of switch B to the desired pattern. *Next,* by Lemma 1, no more than 7 drops in the left hand hole are needed to obtain agreement in switches A, D, and F. And these drops cannot disturb B. *Finally,* by Lemma 2, with no more than 7 drops in the right hand hole, attain agreement in switches C, E, and H. These do not disturb any previously obtained agreements.

So all switches except possibly G can be brought to agree with those of the desired pattern. And this is the theorem.

THEOREM 3. *If one pattern can be obtained from a second, then the second can be obtained from the first.*

Proof. Let it be possible to obtain pattern Γ from pattern Δ. Begin with Δ. Proceed to obtain Γ by marble drops. Using Theorem 2, continue to make drops in order to transform Γ back to either Δ or the pattern in near-agreement with Δ. If it were actually near-agreement, then we would have produced overall a sequence of drops that transformed Δ into a pattern in near-agreement with itself. This is impossible by Theorem 1. Hence the result must actually be Δ.

So the second half of the construction actually did transform Γ into Δ, as required.

6. Equivalent classes of patterns. Theorem 3 shows that all patterns can be separated into distinct, nonoverlapping classes in such a way that any two patterns in the same class can be obtained from one another; and no pattern in another class can be. These classes are termed, in the language of mathematicians, equivalence classes.

We have shown that there are at least two such classes, since a pattern and the one in near-agreement with it are not in the same class, by Theorem 1. We shall now show that there are no more than two classes.

THEOREM 4. *There are only two equivalence classes.*

Proof. Consider any pattern and the one in near-agreement with it. These are in different classes. Now take any third pattern. We wish to show it is in the same class as one of the two already chosen. By Theorem 2, the third pattern can be transformed into one of the first two. And this proves the theorem.

7. Some particular pattern pairs. Our next question will be to determine the conditions under which two patterns actually are in the same equivalence class. For this, we shall require several intermediate results.

LEMMA 3. *If two patterns agree in all except position F, they are in different classes.*

The proof of this follows the lines of Theorem 1. The reader can easily provide the details. By symmetry, the same result holds for switch H.

LEMMA 4. *If two patterns agree in all except position A, they are in different classes.*

This proof is a bit more complicated. Arguing as in Theorem 1, and assuming the two patterns are such that one could be gotten from the other, we specify the number of drops in the left, center and right hand holes as $2k+1$, $2m$, and $2n$. This is so, since A is to be reversed at the end of the operation and hence, must have been through an odd number of changes, while B and C are to be in their original states. Now we must distinguish two cases accordingly as A was originally set left or right.

If it was initially set to the left, then at D and E we have respectively $(k+m)$ and $(m+n)$ actuations. Both these numbers must be even, say $2s$ and $2t$ respectively. And at F, G, and H, we have $[(k+1)+s]$, $(s+t)$, and $(t+n)$ respectively, all of which must be even. It can be shown in familiar fashion as in Theorem 1 that this is impossible, regardless of whether k is assumed even or odd.

On the other hand, if A was originally set to the right, then at D and E, the inputs are $[(k+1)+m]$ and $(m+n)$, which we again set equal to $2s$ and $2t$. The three lower switches are actuated $(k+s)$, $(s+t)$, and $(t+n)$ times respectively, all of which are to be even numbers. Again, the impossibility of this set of conditions follows from either the assumption that k is even or k is odd. Details are left to the reader. The lemma now follows, and applies by symmetry also to switch C.

The reader will also easily be able to provide the proof of the following similar result.

LEMMA 5. *If two patterns agree in all except position B, they are in different classes.*

8. Other pairs of patterns. It remains to consider the switches in the center line, D and E. Here, for a change, we have a different conclusion.

LEMMA 6. *If two patterns agree in all positions except D, they are in the same equivalence class.*

As with Lemma 4, there are two cases to consider. These are similar (but not identical). We shall go through one in detail, leaving the other for the reader. Suppose D is initially set to the left.

The three inputs at the top are as usual $2k$, $2m$, and $2n$. The inputs at D and E are therefore $(k+m)$ and $(m+n)$, which must be odd and even, respectively, say.

(11) $$k + m = 2s + 1,$$

(12) $$m + n = 2t.$$

Since D was initially set left, the three inputs to the bottom switches are

(13) at F, $k + (s + 1)$, even;

(14) at G, $s + t$, even;

(15) at H, $t + n$, even;

(16) Now if k is odd, then

(17) from (11) and (16), m is even;

(18) from (12) and (17), n is even;

(19) from (15) and (18), t is even;

(20) from (14) and (19), s is even.

And (16) and (20) are consistent with (13). Hence $k=1$, $m=n=0$ is a candidate solution. And in fact, since it satisfied (11)–(15), it must meet the required conditions. Therefore an acceptable sequence of marble drops is simply: drop two in the left hand hole. It is immediately clear that this does, in fact, just reverse D. One of the two goes left and reverses A and F, the other goes right to D, then left to F and therefore reverses all three, A, D, and F.

The reader should verify that similar solutions can be found if D is initially set to the right. Having done so, he will have completed the proof of the lemma.

By symmetry, the result of Lemma 6 also applies to switch E.

9. A test for equivalence. An empirical way to test two given patterns to see if one can be obtained from the other is to follow the construction of Theorem 2 to bring them into agreement or near-agreement.

However, for our final theoretical result, we give an easier test involving no experimentation. We first define the *test set* as the set of six switches: A, B, C, F, G, and H.

LEMMA 7. *Two patterns are in the same equivalence class if they agree on the test set. They are in different classes if they differ on exactly one switch in the test set.*

This is essentially another way of saying Theorem 1 and Lemmas 3–6 somewhat more compactly.

LEMMA 8. *Two patterns are in the same equivalence class if the number of switches in the test set on which they disagree is* 2.

Proof. Consider two patterns Δ and Γ that differ on exactly two switches of the test set, say A and B. Let Λ be a pattern that agrees with Γ on A, with Δ on B, and with both elsewhere on the test set. Then Δ is not in the same equivalence class as Λ, by Lemma 7, and likewise Γ is not in the same class as Λ. But there is only one class that Λ does not belong to (since there are only two classes altogether); and we have shown that both Γ and Δ belong to it. The lemma follows.

This leads to:

THEOREM 5. *Two patterns are in the same equivalence class if and only if the number of switches in the test set on which they agree is even* (*namely* 0, 2, 4 *or* 6).

Proof. The case of differing on no switches is Lemma 6. The case of differing on 2 switches is Lemma 8. So now, suppose Γ and Δ differ on four of the test set switches, say A, B, C, and F. Then choose Λ to agree with Δ on A and B; agree with Γ on C and F; and agree with both elsewhere on the test set. Then by Lemma 8, Δ and Λ are in the same class; and Γ and Λ are in the same class. Hence Γ and Δ are in the same class, as required.

This proves the case in which the number of nonmatching switches is 4. The final case, when that number is 6, is handled similarly. An auxiliary pattern is constructed agreeing with one of the given pair on two switches of the test set, and the other on the remaining four. Then the given patterns are both known to be in the same class as the newly constructed one, hence, the same as each other. This completes the proof.

10. Concluding observations. We can recapitulate the theory briefly. All of the possible dot patterns of the eight switches (there are 256 of them) fall into two nonoverlapping classes. In each class, any pattern can be obtained from any other. One way to do this in 15 marble drops or less is described in Theorem 2. If two patterns are in different classes, neither can be obtained from the other. To determine whether two given patterns are in the same class, it suffices to count modulo 2 the number of positions on which they agree in the so-called test set, consisting of the top three and bottom three switches (omitting the middle two).

The richness of the mathematical theory that has been built on so modest a foundation comes as a surprise to the author and may also be unexpected for some readers. There remain many further extensions and unsolved problems. One of the latter of particular interest to puzzlers would be to develop an algorithm for the shortest program (i.e., minimum number of drops) to transform one pattern to another.

SUMS OF SQUARES OF CONSECUTIVE ODD INTEGERS

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Recently several papers have appeared relative to the sums of squares of consecutive integers: [1], [2], [3], [4]. The obvious extension of these investigations is to sums of squares of consecutive terms in arithmetic progressions. This paper will provide theorems pertaining to the special case of arithmetic progressions with difference 2. But inasmuch as the results for even numbers are directly related to those for consecutive integers, this study is limited to the sums of squares of consecutive odd integers, the object being to determine under what conditions such sums are perfect squares.

Notation. The number of consecutive odd integers being considered will be designated by n. The first integer in the sequence will be denoted by x, while the sum of the squares in the case such a sum is a perfect square will be indicated by z^2.

The basic equation is

$$x^2 + (x+2)^2 + (x+4)^2 \cdots + (x+2n-2)^2 = z^2 \tag{1}$$

where x is odd. In summary form this becomes

$$nx^2 + 2n(n-1)x + \frac{2n(n-1)(2n-1)}{3} = z^2. \tag{2}$$

THEOREM 1. *The sum of the squares of n consecutive odd integers cannot be a perfect square unless* $n \equiv 0$, 1, *or* 4, *modulo* 8.

Proof. Since the square of an odd integer is congruent to 1, modulo 8, and since only 0, 1, and 4 are quadratic residues of 8, the result follows immediately from (1).

THEOREM 2. *The sum of the squares of n consecutive odd integers cannot be a perfect square if n is of the form* $3^{2k+1}(3m+1)\lambda^2$, *where* $3m+1$ *is square-free and* $\lambda \not\equiv 0$ (*mod* 3).

Proof. Substituting partially for n in (2), one obtains:

$$\begin{aligned} &3^{2k+1}(3m+1)\lambda^2x^2 + 2\cdot 3^{2k+1}(3m+1)\lambda^2(n-1)x \\ &\qquad + \frac{2\cdot 3^{2k+1}(3m+1)\lambda^2(n-1)(2n-1)}{3} = z^2. \end{aligned} \tag{3}$$

The common factor of the terms on the left-hand side indicates that we may make the substitution $z = 3^k(3m+1)\lambda\, z'$ and transform the equation to

$$3x^2 + 6(n-1)x + 2(n-1)(2n-1) = (3m+1)z'^2. \tag{4}$$

It follows that $2 \equiv z'^2$ (mod 3), so that there can be no solution.

THEOREM 3. *The sum of the squares of n consecutive odd integers cannot be a perfect square if the maximum power of* 3 *dividing n is even and greater than zero.*

Proof. Let n be of the form $3^{2k}(3m \pm 1)$. Then by (2)

$$(5)\qquad 3^{2k}(3m\pm 1)x^2+2\cdot 3^{2k}(3m\pm 1)(n-1)x+\frac{2\cdot 3^{2k}(3m\pm 1)(n-1)(2n-1)}{3}=z^2;$$

then $z=3^kz'$. This transforms (5) to

$$(6)\quad 3(3m \pm 1)x^2 + 6(3m \pm 1)(n - 1)x + 2(3m \pm 1)(n - 1)(2n - 1) = 3z'^2.$$

Since all terms are divisible by 3 except the constant term, no solution is possible.

THEOREM 4. *If $n=2^{2k+1}mt^2$, where $k\geqq 0$, m is square free and $mt\not\equiv 0$ (mod 2), the sum of the squares of n consecutive odd integers cannot be a perfect square.*

Proof. Case 1. $m\not\equiv 0$ (mod 3). Let $z=2^{k+1}mz't$. Then equation (2) becomes

$$(7)\quad 2^{2k+1}mt^2x^2 + 2^{2k+2}mt^2(n - 1)x + \frac{2^{2k+2}mt^2(n - 1)(2n - 1)}{3} = 2^{2k+2}m^2t^2z'^2$$

which can be transformed into

$$(8)\qquad (x + n - 1)^2 + \frac{(n - 1)(n + 1)}{3} = 2mz'^2.$$

Since the constant term and $n-1$ are both odd, x must be even. Accordingly there can be no odd solution.

Case 2. $m\equiv 0$ (mod 3). Let $m=3m'$, where $m'\not\equiv 0$ (mod 3). Then $z=2^{k+1}m'tz'$. Using equation (2) one obtains

$$(9)\qquad 3x^2 + 6(n - 1)x + 2(n - 1)(2n - 1) = 2m'z'^2.$$

To satisfy this relation, x will have to be even, so that an odd solution is impossible.

THEOREM 5. *If $n\not\equiv 0$ (mod 3) and n is square free, no sum of squares of n consecutive odd integers is a perfect square unless 3 is a quadratic residue of every factor of n.*

Proof. The general equation (2) can be transformed by the relation $z=nz'$ to

$$(10)\qquad 3(x + n - 1)^2 + (n - 1)(n + 1) = 3nz'^2$$

so that $3(x+n-1)^2\equiv 1$ (mod n). Thus, unless 3 is a quadratic residue of each factor of n, the product of a nonresidue and a residue would be a residue, which is impossible.

Primes for which 3 is a nonresidue are of the form $12k+5$ or $12k+7$; thus the presence of such factors in n indicates that there can be no solution.

THEOREM 6. *If n is of the form t^2m or $3^{2k+1}t^2m$, where m is square free and not divisible by 3, it is not possible to have the sum of the squares of n consecutive odd integers equal to a perfect square if m has a prime factor of the form $12k+5$ or $12k+7$.*

This theorem generalizes Theorem 5 and is similarly proved.

THEOREM 7. *If n is square free and of the form $12m+1$, while $6m+1$ has a factor of the form $4k+3$, there cannot be n consecutive odd integers whose sum of squares is a perfect square.*

Proof. Substituting into (2) we obtain

$$(12m + 1)x^2 + 2(12m + 1)(12m)x + \frac{2(12m + 1)(12m)(24m + 1)}{3} = z^2. \tag{11}$$

Let $z = (12m+1)z'$. Then

$$(x + 12m)^2 + 8m(6m + 1) = (12m + 1)z'^2 \quad \text{and} \tag{12}$$
$$(x + 12m)^2 \equiv -z'^2 \pmod{6m + 1}.$$

Therefore if $6m+1$ has a prime factor of the form $4k+3$, there is a relation: a residue is the product of a nonresidue and a residue, which is impossible.

The application of these theorems in specific cases is indicated for values of n up to 241 in the following table. The number opposite n specifies the theorem which eliminates the possibility of there being n consecutive odd integers with the sum of their squares a perfect square. In case none of the theorems eliminate the value of n but some special considerations do, an x is used. S means that a solution exists. A blank space denotes an unresolved case. It should be noted likewise that in many instances the value of n is eliminated by more than one theorem, but this point is not given consideration in the table.

TABLE I. SUM OF THE SQUARES OF n CONSECUTIVE ODD INTEGERS.
VALUES OF n AS RELATED TO THEOREMS

n	Th.	n	Th.	n	Th.	n	Th.	n	Th.
4	x	52	S	100	S	148	S	196	S
8	4	56	4	104	4	152	4	200	4
9	3	57	2	105	6	153	3	201	2
12	2	60	6	108	2	156	2	204	6
16	S	64	S	112	6	160	4	208	x
17	5	65	5	113	5	161	5	209	5
20	6	68	6	116	6	164	6	212	6
24	4	72	4	120	4	168	4	216	4
25	S	73	S	121	S	169	S	217	5
28	6	76	6	124	6	172	6	220	6
32	4	80	6	128	4	176	x	224	4
33	S	81	3	129	2	177	S	225	3
36	3	84	2	132	x	180	3	228	2
40	4	88	4	136	4	184	4	232	4
41	5	89	5	137	5	185	5	233	5
44	x	92	x	140	6	188	x	236	x
48	2	96	4	144	3	192	2	240	6
49	S	97	x	145	5	193		241	7

SOLUTIONS. In cases in which it was impossible to exclude a given value of n, solutions were sought. In very many instances, values were found which led to a negative solution for the initial x. In such cases it was necessary to use a rather involved procedure which often led to the extremely large quantities found in the table of solutions. The process will be illustrated for the case of 996 consecutive odd integers.

Substituting $n=996$ into the basic equation (2), one obtains

$$996x^2 + 2\cdot 996\cdot 995x + \frac{2\cdot 996\cdot 995\cdot 1991}{3} = z^2.$$

The substitution $z=166z'$ leads to

$$3(x + 995)^2 + 995\cdot 997 = 83z'^2.$$

Let $x'=x+995$. Then $x'^2 \equiv 28 \equiv 32^2 \pmod{83}$, so that $x' \equiv \pm 32 \pmod{83}$. Let $x'=83\lambda \pm 32$. Then

$$249\lambda^2 \pm 192\lambda + 11989 = z'^2.$$

For the positive sign, $\lambda=10$, $z'=197$ constitutes a solution. Unfortunately, $x'=830+32=862$ and $x=862-995$ so that a negative value is obtained for x.

To find another solution, the related Pell equation $249p^2+1=q^2$ can be solved. This is done by the usual continued fraction method leading to values $p=9{,}273{,}635{,}639{,}880$ and $q=146{,}335{,}502{,}108{,}449$. Setting $\lambda_1=10$, $z_1'=197$, additional solution values can be found from the relations due to Euler [5]:

$$\lambda_2 = q\lambda_1 + z_1' p + b(q - 1)/2a, \qquad z_2' = ap\lambda_1 + z_1' q + bp/2,$$

where a is the coefficient of the square term and b of the first-degree term in the equation being solved. Thus in the present instance,

$$\lambda_2 = 10q + 197p + \frac{192(q - 1)}{2\cdot 249},$$

$$x_2 = x_2' - 995 = 83\lambda_2 + 32 - 995 = 862q + 16351p - 995,$$

$$x_2 = 277{,}774{,}419{,}165{,}159{,}923,$$

$$z_2 = 166z_2' = 166(2490p + 197q + 192p/2),$$

$$z_2 = 429276p + 32702q = 8{,}766{,}412{,}802{,}895{,}626{,}078.$$

If $a'=(x_2-1)/2$, the sum of the squares of the 996 consecutive odd integers beginning with x_2 is given by

$$\sum_{a'+1}^{a'+996} (2k - 1)^2 = \frac{(a' + 996)(2a' + 1991)(2a' + 1993)}{3} - \frac{a'(2a' - 1)(2a' + 1)}{3}$$
$$= 3{,}984a'^2 + 3{,}968{,}064a' + 1{,}317{,}396{,}916$$

which equals 76,849,993,430,772,347,036,770,647,183,593,662,084. This checks with the square of z_2.

For $n \leqq 1000$, there remain the following unresolved cases: $n=193, 564, 577, 601, 673, 724, 772$, and 913.

TABLE 2. SOLUTION VALUES FOR n CONSECUTIVE ODD INTEGERS WHOSE SUM OF SQUARES IS A PERFECT SQUARE

n	x	z
16	27	172
25	27	265
33	1517	8899
49	151	1407
52	225	2002
64	619	5464
73	517495600589935	4421484349626365
100	1567	16670
121	1099	13431
148	8849	109446
169	2211	30953
177	1199131971875	15953413274027
196	6207	89642
249	43768494276322787	690655190173008145
256	10667	174768
289	6671	118337
292	1158764983	19800989682
297	4421	81345
313	2637	52271
337	2899539686301218938577	53228473040759443442535
361	10499	206359
388	38107233499	750625455526
393	239273212453493311	47434066226438066315
400	26267	533340
409	19483668275782587481247129	394032805433280341208689181
457	105	13253
481	43	12987
484	38559	858946
529	22791	536383
537	76577831996932535255	1774558045753696372163
592	50205	1235948
625	31927	813825
628	18089637912041409	453325026738386914
649	1154223762921849084337214263210827	29404402547677514405437753288098735
676	75487	1980238
708	4396701331467	116988613479574
753	106077943132957107231808597	2910868445077365235846422131
784	101659	2868404
793	2357	89609
841	58099	1709289
852	86179493467053	2515496599165978
897	1187	64285
961	75999	2385791
964	7224941010476874595	224322492833632877066
976	1317265505365	41152689975900
996	277774419165159923	8766412802895626078

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1. H. L. Alder, n and $n+1$ consecutive integers with equal sums of squares, Amer. Math. Monthly, 69 (1962) 282–285.
2. Brother U. Alfred, n and $n+1$ consecutive integers with equal sums of squares, this MAGAZINE 35 (1962) 155–164.
3. ———, Consecutive integers whose sum of squares is a perfect square, this MAGAZINE, 37 (1964) 19–32.
4. Stanton Philipp, Note on consecutive integers whose sum of squares is a perfect square, this MAGAZINE, 37 (1964), 218–220.
5. L. E. Dickson, History of the Theory of Numbers, vol. 2, Chelsea, New York, 1952. p. 354.

ANSWERS

A412. It has often been assumed that the solution leads to three conditions, namely (1) $q(x)<0$, (2) $-a^2q(x)>p(x)$, and (3) $p(x)>a^2q(x)$. However, conditions (2) and (3) imply (1) for $a^2q(x)<-p(x)$ and $a^2q(x)<p(x)$. Adding these inequalities gives $2a^2q(x)<0$ which is equivalent to (1). A similar argument holds when $q(x)>0$.

A413. The statement is true and follows from a result of E. W. Chittenden (Note on functions which approach a limit at every point of an interval, *American Mathematical Monthly*, 25 (1918) 249) which states that under the hypotheses of the statement, the set of discontinuities of f is countable.

A414. Now p must be of the form 3, $3m+1$, or $3m+2$, while d must be of the form $3n+1$ or $3n+2$. Going through the six possibilities, we find there is only one prime, $p=3$.

A415. If the argument of the characteristic function of A is set equal to c, adding the first $n-1$ rows (columns) to the nth row (column) yields a determinant whose nth row (column) is zero. Thus c is a characteristic root of A.

(Quickies on page 232)

ERRATA. January, 1967, Page 30. The solution A 400 should read: "The only solution is $y=0$ since $D^nx^{2n}D^n\equiv x^nD^{2n}x^n$. This follows from $D^mx^m=x^mD^m+a_1x^{m-1}D^{m-1}+\cdots+a_m$ by Leibniz' Theorem and $x^rD^r=xD(xD-1)\cdots(xD-r+1)\cdots$".

March, 1967, Page 108. Parts (a) and (b) were omitted from the statement of Problem 611. Page 109, Part (b), Line 2, at the end a denominator 2 is omitted. Page 109, bottom, the first column of the table should read: "x, w, l, W, L." Page 110, first line, the numeral "1" should be the letter "l".

M. S. Klamkin found Problem 626 [May, 1966, and January, 1967] as a theorem in R. A. Johnson, *Advanced Euclidean Geometry*.

References

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SOLUTION OF CERTAIN LINEAR BOUNDARY VALUE PROBLEMS BY USE OF FINITE TRANSFORMS

JOHN W. CELL, North Carolina State University

Methods of finite transforms apply to the same group of problems as does ordinary separation of variables. However, use of finite transform methods expedites the solution, particularly where tables of transforms are available. References for explanation of finite transform methods are given in the bibliography at the end of this note and include the books by Churchill [**1**], Sneddon [**2**], and Tranter [**3**].

The more commonly used finite transforms are listed below. In three cases we also supply the basic transform relation.

I. Finite sine transform:

$$f_s(n) = S\{F(x)\} = \int_0^\pi F(x) \sin nx \, dx, \qquad n = 1, 2, \cdots$$

$$F(x) = (2/\pi) \sum_1^\infty f_s(n) \sin nx$$

$$S\{F''(x)\} = n\{F(0+) - (-1)^n F(\pi-)\} - n^2 S\{F(x)\}$$

See Churchill, [**1**], page 277, for table.

II. Finite cosine transform:

$$f_c(n) = C\{F(x)\} = \int_0^\pi F(x) \cos nx \, dx, \qquad n = 0, 1, 2, \cdots$$

$$F(x) = (1/\pi) f_c(0) + (2/\pi) \sum_1^\infty f_c(n) \cos nx$$

$$C\{F''(x)\} = -n^2 C\{F(x)\} - F'(0+) + (-1)^n F'(\pi-).$$

See Churchill, [**1**], page 278 for table.

III. Special trigonometric transform:

$$f(p) = \int_0^1 F(x) \cos px \, dx$$

where $p_1, p_2, \cdots$ are solutions of $p \tan p = h$, and where

$$F(x) = 2 \sum_p \frac{p^2 + h^2}{p^2 + h^2 + h} f(p) \cos px.$$

IV. Finite Hankel transform:

$$f(p) = \int_0^1 F(r) r J_n(pr) dr$$

where $p_1, p_2, \cdots$ are the positive roots of $J_n(p) = 0$. Then

$$F(r) = 2 \sum_{p} f(p) [J_n(pr)/J_{n+1}^2(p)].$$

Note:

$$\int_0^1 \frac{d}{dr} [rF'(r)] J_0(pr) dr = + pF(1-) J_1(p) - p^2 f(p).$$

See Sneddon, [2], page 531, for brief table.

Other transform definitions could be generated for each set of orthogonal polynomials as, for example: Legendre, Tschebyscheff, Laguerre, and Hermite polynomials. To illustrate this idea we refer to Laguerre polynomials defined for nonnegative integers n by

$$L_n(x) = 1 - \binom{n}{1} x + \binom{n}{2} \frac{x^2}{2!} - \binom{n}{3} \frac{x^3}{3!} + \cdots.$$

These polynomials satisfy the differential equation

$$xy'' + (1 - x)y' + ny = 0,$$

and the recursion relation

$$nL_n(x) = (2n - x - 1) L_{n-1}(x) - (n - 1) L_{n-2}(x).$$

If one defines

$$f(n) = \int_0^\infty e^{-x} F(x) L_n(x) dx, \qquad n = 0, 1, 2, \cdots$$

for an appropriate class of functions $F(x)$, then

$$F(x) = \sum_{n=0}^{\infty} f(n) L_n(x).$$

These last two relations constitute the definition of the Laguerre transform. The basic transform relation is given by

$$\int_0^\infty e^{-x} [xF''(x) + (1 - x)F'(x)] L_n(x) dx = - nf(n).$$

Laguerre expansions are of special interest through the Laplace transform relations:

$$L\{e^{-t} L_n(t)\} = s^n/(s + 1)^{n+1}$$
$$L\{e^{-t/2} L_n(t)\} = 2(2s - 1)^n/(2s + 1)^{n+1}.$$

See, for example, pages 237–240 of the chapter by Sneddon [4].

The decision as to which transform (or set of transforms) to use in a given boundary-value problem depends on the structure of the operator involving one of the independent variables, the domain of that variable, and last but not least, on experience. Sometimes the domain can be modified or the structure of the operator changed by an appropriate transformation. We proceed to illustrate with some examples.

Find the solution of the following boundary-value problem:

$$U_{xx}(x, y) + U_{yy}(x, y) = 0, \qquad 0 < x < \pi, \qquad 0 < y < \pi; \tag{1}$$

$$U(0, y) = 0, \qquad U(\pi, y) = 1, \qquad 0 < y < \pi; \tag{2}$$

$$U(x, 0) = 0, \qquad U(x, \pi) = 0, \qquad 0 < x < \pi. \tag{3}$$

The fact that only even order derivatives in x and in y appear, that the domain in each case is from 0 to π, and that the boundary conditions are on the function and not its first partial derivative suggests the use of a finite sine transform with respect to x or y (or both). We illustrate:

(a) We define $u(n, y) = S\{U(x, y)\}$, take the finite sine transform with respect to x and obtain formally:

$$\begin{aligned} (d^2u/dy^2) - n^2u &= n(-1)^n, \\ u(n, 0) = 0, \qquad u(n, \pi) &= 0. \end{aligned} \tag{4}$$

The solution of this two-point boundary-value problem is easily found by elementary methods to be

$$u(n, y) = \frac{(-1)^{n-1}}{n}\left\{1 - \frac{\sinh ny}{\sinh n\pi} - \frac{\sinh n(\pi - y)}{\sinh n\pi}\right\}. \tag{5}$$

The inverse sine transform relation yields the solution

$$U(x, y) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\sin nx}{n}\left\{1 - \frac{\sinh ny}{\sinh n\pi} - \frac{\sinh n(\pi - y)}{\sinh n\pi}\right\}. \tag{6}$$

(b) Suppose that, instead of using classical methods to solve the two-point boundary value problem in (a), we define $v(n, m) = S\{u(n, y)\}$, take the sine transform with respect to y, and obtain

$$v(n, m) = \frac{n(-1)^{n-1}}{m(n^2 + m^2)}\{1 - (-1)^m\}.$$

The double inversion yields

$$U(x, y) = \frac{4}{\pi^2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{(-1)^{n-1}n\{1 - (-1)^m\}}{m(n^2 + m^2)}\sin my \sin nx. \tag{7}$$

(c) As a third solution we define

$$w(x, m) = S\{U(x, y)\};$$

take the sine transform with respect to y of the original boundary-value problem, and obtain

$$\begin{aligned} (d^2w/dx^2) - m^2w &= 0, \\ w(0, m) &= 0, \\ w(\pi, m) &= \{1 - (-1)^m\}/m. \end{aligned} \tag{8}$$

The solution is easily obtained and the inverse sine transform found to be

$$U(x, y) = \frac{4}{\pi} \sum_{m=1,3,5,\cdots} \frac{\sinh mx \sin my}{m \sinh m\pi}. \tag{9}$$

This third solution is the simplest. The double-series solution obtained in (7) is not as useful as are the other two but really is equivalent to the other two solutions and involves Fourier sine series expansions of an appropriate function appearing in each of the other two solutions.

If one uses the usual method of separation of variables on (1), one forms $U = X(x)\ Y(y)$ and is led to

$$X'' + k^2X = 0, \qquad Y'' - k^2Y = 0; \tag{10}$$

or

$$X'' - m^2X = 0, \qquad Y'' + m^2Y = 0; \tag{11}$$

or

$$X'' = 0, \qquad Y'' = 0.$$

If one proceeds from (11) assuming m to be real, one is led to the third solution given in (9).

However, let us pursue the equation in (10) and assume that k is real. Then $X = a \cos kx + b \sin kx$, $Y = A \cosh ky + B \sinh ky$ and $U(x, y) = X(x)\,Y(y)$. Since $U(0, y) = 0$ and $U(x, 0) = 0$ we are led to $U(x, y) = A \sin kx \sinh ky$. But the other two conditions, $U(\pi, y) = 1$ and $U(x, \pi) = 0$, lead to impossibilities, and a series of such terms

$$U(x, y) = \sum_{n=1}^{\infty} A_n \sin nx \sinh ny \tag{12}$$

is likewise not appropriate.

Now the structure of the solution (6) suggests that we are seeking a function

$$U(x, y) = \frac{2}{\pi} \sum_{n=1}^{\infty} \phi_n(y) \sin nx, \qquad 0 < x < \pi, \qquad 0 < y < \pi. \tag{13}$$

Then, formally, from the usual rules for Fourier series

$$\phi_n(y) = \int_0^{\pi} U(x, y) \sin nx dx. \tag{14}$$

The second relation in (10) suggests that we consider

$$\phi_n''(y) - n^2\phi_n(y) = \int_0^{\pi} \{U_{yy} - n^2U\} \sin nx dx.$$

This, by use of the Laplacian (1), becomes

$$-\int_0^{\pi} \{U_{xx} + n^2U\} \sin nx dx$$

By repeated integration-by-parts and the assumption that $U(x, y)$ and $U_x(x, y)$ are continuous in x in $0<x<\pi$, we obtain

$$\phi_n''(y) - n^2\phi_n(y) = -n\{U(0+, y) - (-1)^n U(\pi-, y)\}. \tag{15}$$

This reduces to $\phi_n''(y) - n^2\phi_n(y) = 0$ only if the right hand member of (15) is zero. (We observe that (15) is in agreement with (4) and is in fact the sine transform of (1).) What is now to be made clear is that the assumed Fourier sine series in (13) is for each y to have $U(\pi-, y) = 1$. This can be satisfied only if we allow $U(x, y)$ to have a finite jump at $x = \pi$. Note especially that we are using $U(\pi-, y) = 1$ and *not* $U(\pi, y)$, which is zero.

Thus the use of the sine transform leads to a somewhat larger domain of functions for solution than would the ordinary method of separation of variables. There is, incidentally, nothing restrictive in the example employing the Laplacian. We could have used a wave or heat equation in two independent variables to gain the same type of result in (15).

As a second example let us find the formal solution of the "heat" problem:

$$U_t(r, t) = U_{rr}(r, t) + (1/r)U_r(r, t), \quad 0 \leqq r < 1, t > 0; \tag{16}$$

$$U(r, 0+) = 0, \quad 0 \leqq r < 1; \tag{17}$$

$$U(1, t) = 1 \quad \text{and} \quad |U(0, t)| < \infty \quad \text{for } t > 0. \tag{18}$$

The form of the problem suggests the possible use of the Laplace transform with respect to t or the finite Hankel transform with respect to r or the two together. We use the finite Hankel transform and define

$$u(p, t) = H\{U(r, t)\} = \int_0^1 U(r, t) r J_0(pr) dr.$$

Then, formally, the Hankel transformed problem reads as follows:

$$du/dt = -p^2 u + pJ_1(p), \quad u(p, 0+) = 0.$$

The solution found by elementary methods is

$$u(p, t) = \{J_1(p)/p\}\{1 - e^{-p^2 t}\}.$$

The inverse Hankel transform yields

$$U(r, t) = 2 \sum_p \frac{J_0(pr)}{pJ_1(p)} \{1 - e^{p^2 t}\} \tag{19}$$

where the summation is on the ordered positive zeros of $J_0(p) = 0$, namely $0 < p_1 < p_2 < p_3 < \cdots$

We observe that formally $U(1, t) = 0$ whereas $U(1, t)$ is supposed to be 1. We may rewrite (19) for $0 \leqq r < 1$ and $t > 0$ as

$$U(r, t) = 1 - V(r, t) \tag{20}$$

since

$$2\sum_{p} \frac{J_0(pr)}{pJ_1(p)} = 1 \quad \text{for } 0 \leqq r < 1.$$

Also $V(r, t)$ is the solution of the related boundary-value problem:

$$V_t(r, t) = V_{rr}(r, t) + (1/r)V_r(r, t), \quad 0 \leqq r < 1, \quad t > 0,$$

$$V(r, 0+) = 1,$$

$$V(1, t) = 0 \quad \text{and} \quad V(0, t) = \text{finite for } t > 0.$$

The assumed inconsistency is now explained; the series in (19) for $0 \leqq t$ converges but not uniformly in $1-\epsilon \leqq r \leqq 1$ and there is a jump at $r=1$.

If one uses the method of separation of variables on the given heat-problem, one obtains

$$\frac{T'}{T} = \frac{rR'' + R'}{R} = C = \text{constant}.$$

To effect the complete solution a combination of values for C must be employed —both zero and negative values. Put differently, one obtains the form given in (20) and solves for $V(r, t)$ by the usual separation of variables method.

If in the given equation (16) one employs the Laplace transform with respect to t, one obtains

$$s\bar{u} - \frac{d^2\bar{u}}{dr^2} + \frac{1}{r}\frac{d\bar{u}}{dr},$$

$$\bar{u}(1, s) = \frac{1}{s}, \quad \bar{u}(0, s) = \text{finite}, \quad \text{where } \bar{u}(r, s) = L\{U(r, t)\}.$$

The solution by elementary methods is

$$\text{(21)} \qquad \bar{u}(r, s) = \frac{1}{q^2}\frac{I_0(qr)}{I_0(q)}$$

where $q^2 = s$. One form of inverse Laplace transform is readily found to be that already given in (19). Another form especially useful if t is "small" is found by use of the asymptotic expansion of $I_0(z)$:

$$I_0(z) \sim \{e^z/\sqrt{2\pi z}\}\left\{1 + \frac{1}{8z} + \frac{9}{128z^2} + \cdots\right\}$$

whence

$$\bar{u} \sim \frac{1}{s\sqrt{r}}\, e^{-(1-r)q}\left\{1 + \frac{1-r}{8qr} + \frac{9 - 2r - 7r^2}{128q^2r^2} + \cdots\right\}$$

and, formally,

$$U(r, t) = \frac{1}{\sqrt{r}}\operatorname{erfc}\left\{\frac{1-r}{z\sqrt{r}}\right\} + \frac{1-r}{4r}\{\sqrt{t}\}\operatorname{ierfc}\left\{\frac{1-r}{2\sqrt{t}}\right\} + \cdots.$$

See Tranter, [3], page 28 or S. Goldstein, [5].

We note that the Hankel finite transform method is much simpler than the Laplace transform method to secure the result of (19) or (20). The Laplace transform method, however, offers some choice in the form and useful domain of the inverse transform.

The results presented in this note were obtained as a part of research in the Applied Mathematics Research Group at North Carolina State University sponsored by AFOSR, ARO, and ONR through the Joint Services Advisory Group. The current grant is AFOSR-444-64.

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THE 202 OCTAGONS

BRUCE L. CHILTON, State University of New York at Buffalo

Select a vertex of the regular octagon; label it 0; and proceeding counterclockwise, label the others 1, 2, $\cdots$, 7 consecutively. A permutation $a_1a_2a_3 \cdots a_7$ of the first seven positive integers will then designate the octagon obtained by joining vertex 0 to vertex a_1, a_1 to a_2, $\cdots$, a_6 to a_7, and a_7 to 0. (Thus, in particular, 1234567 is the regular octagon itself.)

We define two octagons of the set obtainable in this manner to be *equivalent* if they are related by an isometry of the plane, and *distinct* if they are not. S. Golomb and L. Welch enumerated the distinct forms, arriving at the number 202. (They give in addition the number of forms obtained if we consider two forms equivalent only when they are related by a *direct* isometry, i.e., a rotation or a translation.) The present paper illustrates the 202 distinct octagons and gives details as to how they were found.

Of the $7!=5040$ permutations of the numbers 1, 2, $\cdots$, 7, it is sufficient to consider half, since it is evident that the permutation $a_7a_6a_5 \cdots a_1$ will represent an octagon not only equivalent to $a_1a_2a_3 \cdots a_7$, but in the same position. We thus consider only those permutations of which the last digit exceeds the first.

The problem is to sort the permutations into equivalence classes. We have to ask: How are the permutations affected when various isometries are applied to the corresponding octagons? We need only consider isometries which leave the center of the octagon invariant, namely, rotations and reflections.

The permutation $a_1a_2 \cdots a_7$ can be written $0a_1a_2 \cdots a_7$, where 0 is the "initial vertex." Any ordered 8-tuple cyclically related to this one, such as

See Tranter, [3], page 28 or S. Goldstein, [5].

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The permutation $a_1a_2 \cdots a_7$ can be written $0a_1a_2 \cdots a_7$, where 0 is the "initial vertex." Any ordered 8-tuple cyclically related to this one, such as

$a_5a_6a_70a_1a_2a_3a_4$, will represent the same octagon, in the same position; we have merely chosen a new initial vertex. Similarly, a reversal of the digits, $a_7a_6 \cdots a_10$, will not affect the octagon in any way; nor will any combination of the operations of cyclic shifting and reversal. Using these operations, we can always put a permutation of the numbers $0, 1, \cdots, 7$ into the form $0a_1a_2 \cdots a_7$, with $a_7 > a_1$.

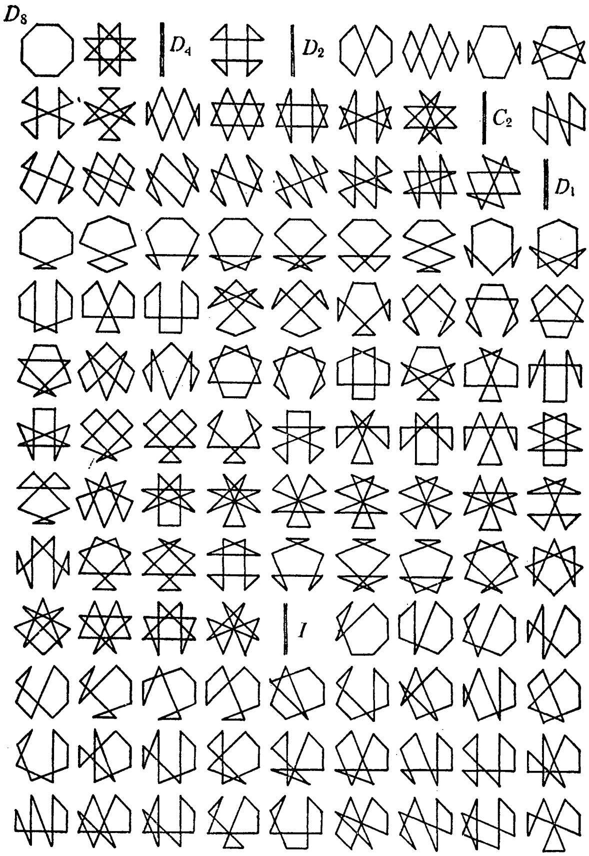

PLATE 1.

(i) *Rotations.* A counterclockwise rotation through $2\pi r/8$ radians corresponds to the addition (modulo 8) of r to each digit of $0a_1a_2 \cdots a_7$. For example, suppose we begin with the permutation 2375416. Writing this as 02375416, and adding 3 (modulo 8), we obtain 35620741, which can be written as 2653147.

(ii) *Reflections.* Reflection in the line 04 corresponds to replacing $0a_1a_2 \cdots a_7$ by $0b_1b_2 \cdots b_7$ where $a_i + b_i \equiv 0 \pmod 8$ $(i = 1, 2, \cdots, 7)$. Any other reflection can be obtained as the product of this reflection with a rotation.

Sorting the 2520 permutations of $1, 2, \cdots, 7$ which have the first digit less than the last into equivalence classes is effected by a sieving technique, applying the operations of (i) and (ii) in all possible ways to a given permutation, crossing the resulting permutations off the list, and continuing with the remaining list.

The number of permutations in an equivalence class varies according to the degree of symmetry of the octagon corresponding to it. The regular octagon itself represents the highest possible degree of symmetry, and its equivalence class contains only one permutation, 1234567. Only one other octagon has such a high degree of symmetry, namely the "stellated" regular octagon, formed by joining each vertex, a, (where $a=0, 1, 2, \cdots, 7$) to vertex $a+3$ (modulo 8). The permutation of this is 3614725. Note that it is impossible to form an octagon in which each vertex a is joined to $a+2$; instead, we would get two squares.

On the other hand, completely asymmetrical octagons have 16 permutations in their equivalence classes. The equivalence class of an octagon whose symmetry group is of order k contains $16/k$ permutations.

The symmetry groups represented are the dihedral groups of orders 16, 8, 4, and 2, named respectively D_8, D_4, D_2, and D_1; the cyclic group C_2; and the identity group I. The number of octagons in each case is respectively 2, 1, 11, 58, 9, and 121.

Although this paper is exclusively concerned with the octagon, the techniques can readily be adapted to n-gons in general. Results for the cases $n=3, \cdots, 7$ are illustrated in the paper of Golomb and Welch. For $n>8$, the large number of permutations which must be sieved necessitates the use of a computer. The author has, with the assistance of M. Wunderlich, recently obtained from the IBM 7044 the 1219 distinct figures having the vertices of the regular 9-gon.

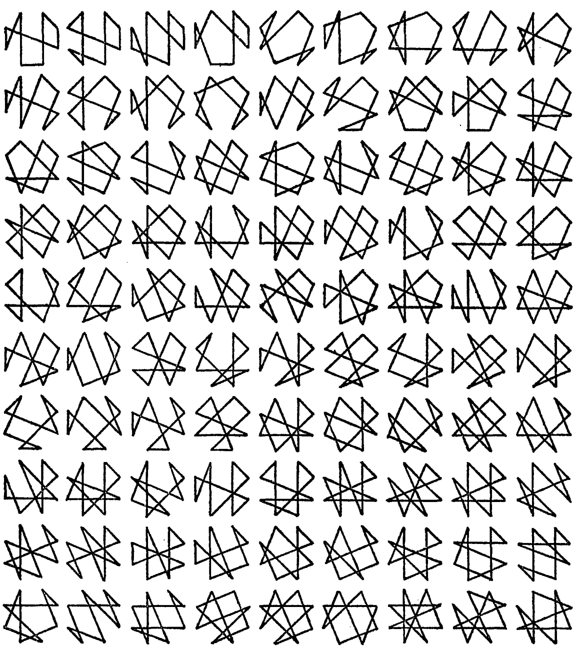

PLATE 2.

Reference

1. S. W. Golomb and L. R. Welch, On the enumeration of polygons, Amer. Math. Monthly, 67 (1960) 349–353.

THE REPRESENTATION OF A GAUSSIAN INTEGER AS A SUM OF TWO SQUARES

L. J. MORDELL, St. Johns College, Cambridge, England

Niven [1] has proved the

THEOREM. *If a and b are rational integers, a representation $x^2+y^2=a+2ib$ in Gaussian integers is possible if and only if not both $a\equiv 2 \pmod 4$, $b\equiv 1 \pmod 2$ are satisfied.*

A simple proof has been given recently by Leahey [2]. A simpler version is as follows. Write

$$(x+iy)(x-iy) = a+2ib.$$

If a is odd, a solution is given by

$$x+iy = a+2ib, \qquad x-iy = 1.$$

Suppose next that a is even, say $a=2a_1$, so that

$$(x+iy)(x-iy) = 2a_1+2ib. \tag{1}$$

If a_1 and b are both even, a solution is given by

$$x+iy = a_1+ib, \qquad x-iy = 2.$$

If a_1 and b are of different parity, a solution is given by

$$x+iy = (1+i)(a_1+ib), \qquad x-iy = 1-i,$$

since

$$a_1 - b + 1 \equiv 0 \pmod 2.$$

If $a_1\equiv b\equiv 1 \pmod 2$, the representation is impossible.

For if $x=X+iY$, $y=Z+iW$, then from (1),

$$X^2-Y^2+Z^2-W^2 \equiv 2 \pmod 4, \qquad XY+ZW \equiv 1 \pmod 2.$$

Hence either $ZW\equiv 1 \pmod 2$ and then $X^2-Y^2\equiv 2 \pmod 4$, or $XY\equiv 1 \pmod 2$ and then $Z^2-W^2\equiv 2 \pmod 4$, and these are impossible.

References

1. Ivan Niven, Integers of quadratic fields as sums of squares, Trans. Amer. Math. Soc., 48 (1940) 405–417.
2. W. J. Leahey, A note on a theorem of I. Niven, Proc. Amer. Math. Soc., 16 (1965) 1130–1131.

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References

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ON THE LIMIT POINTS OF THE SEQUENCE $\{\sin n\}$

JOHN H. STAIB and MILTIADES S. DEMOS, Drexel Institute of Technology

Suppose that the points 1, 2, 3, $\cdots$ are "projected" onto the y-axis by the graph of $y=\sin x$. (See Figure 1.) Then it is intuitively evident that no subinterval of $[-1, 1]$ will elude these particles. Or, more precisely, every point in the interval $[-1, 1]$ is a limit point of the sequence $\{\sin n\}$.

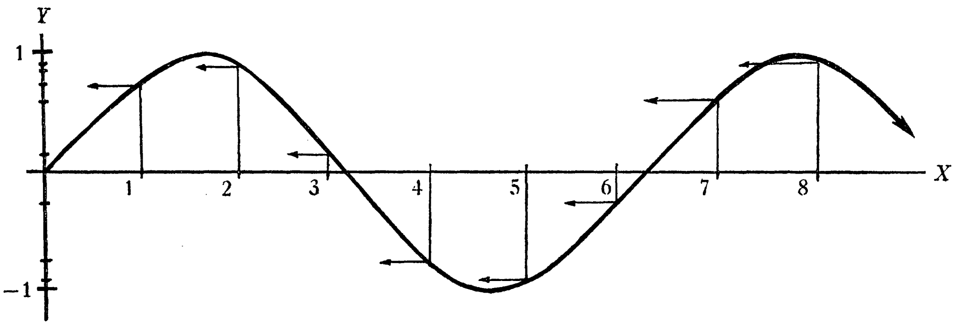

FIG. 1.

This proposition is true. Indeed, it is just one of a family of analogous propositions. And, as is frequently the case with such families, one member is easier to prove than all the others. Our plan here is to first prove this "easy" case and then show how that result can be extended.

We introduce the following notation: $(x)=x-[x]$, where $[x]$ is the greatest integer in x. (Evidently, $0\leqq(x)<1$.) We shall show that every point in $[0, 1]$ is a limit point for the sequence $\{(n\alpha)\}$, provided that α is irrational. But first, we need to know certain properties of the function (x); they arise as corollaries to the following theorem.

THEOREM 1.

$$(x+y)=\begin{cases}(x)+(y), \text{ if } (x)+(y)<1\\(x)+(y)-1, \text{ if } (x)+(y)\geqq 1.\end{cases}$$

Proof. It is immediate from the definition of (x) that $0\leqq(x)+(y)<2$. We suppose first that

$$0\leqq(x)+(y)<1.$$

Equivalently, we have

$$-x-y\leqq-[x]-[y]<1-x-y,$$
$$x+y-1<[x]+[y]\leqq x+y.$$

It follows that $[x]+[y]=[x+y]$. Therefore

$$(x+y)=x+y-[x+y]=x-[x]+y-[y]=(x)+(y).$$

The alternate result is derived in identical fashion, starting with $1\leqq(x)+(y)<2$.

PROPERTY 1. *For* $x \neq$ *integer*, $(-x) = 1 - (x)$.

Proof. Writing $-x$ for y, we have

$$(0) = (x) + (-x) + \{0 \text{ or } -1\}.$$

But (x) and $(-x)$ are both positive.

PROPERTY 2. *If* $(z) > (x)$, *then* $(z-x) = (z) - (x)$.

Proof. Writing $(z-x)$ for y, we have

$$(z) = (x) + (z - x) + \{0 \text{ or } -1\}.$$

But $(z) - (x) = (z-x) - 1 < 0$ denies our hypothesis.

PROPERTY 3. *If* $n(x) < 1$, *where* n *is a natural number, then* $(nx) = n(x)$.

Proof. By induction.

LEMMA. *Given* $\epsilon > 0$ *and an irrational number* α, *there exists a natural number* n *such that* $(n\alpha) < \epsilon$.

Proof. (We may assume that $\epsilon < 1$.) Choose N such that $N > 1/\epsilon$, and consider the set

$$R = \{(\alpha), (2\alpha), (3\alpha), \cdots, (N\alpha)\}.$$

Letting $b = \max R$, we see that R partitions $[0, b]$ into N subintervals. Moreover, the smallest such subinterval must be of length not exceeding b/N. In other words, there exist distinct nonnegative integers k and j such that

$$0 < (k\alpha) - (j\alpha) \leqq b/N < 1/N < \epsilon.$$

(If the smallest subinterval is to the far left, we take $k = 1$ and $j = 0$.) It follows, using Property 2, that

$$0 < (\{k - j\}\alpha) < \epsilon.$$

Now it may happen that $k - j < 0$. (Otherwise, we are finished.) In this event we let $-m = k - j$ and appeal to the following argument: Since $(-m\alpha) < \epsilon$, we may write $-m\alpha = [-m\alpha] + (m\alpha)$, thus,

$$-m\alpha = \text{negative integer} + \epsilon^*,$$

where $0 < \epsilon^* < \epsilon < 1$. Next, multiply both sides by p, where p is the largest natural number such that $p\epsilon^* < 1$. We obtain

$$-pm\alpha = \text{negative integer} + p\epsilon^*.$$

Thus, $(-pm\alpha) = p\epsilon^*$. And also, by our choice of p, we are assured that $0 < 1 - p\epsilon^* < \epsilon^*$. Therefore,

$$0 < 1 - (-pm\alpha) < \epsilon^* < \epsilon.$$

Finally, applying Property 1, we have

$$0 < (pm\alpha) < \epsilon.$$

Thus, we may take $n = pm$.

THEOREM 2. *Let α be irrational. If u is in $[0, 1]$, then u is a limit point for the sequence $\{(n\alpha)\}$.*

Proof. We take u in $(0, 1]$ and immediately apply our lemma: Given $\epsilon > 0$, but less than u, choose a natural number k such that $(k\alpha) < \epsilon$. Then take j as the natural number for which

$$j(k\alpha) \leqq u < j(k\alpha) + (k\alpha).$$

It follows that

$$0 \leqq u - j(k\alpha) < (k\alpha) < \epsilon.$$

Since $u < 1$, these inequalities imply that $j(k\alpha) < 1$. Thus Property 3 is applicable; we may write

$$0 \leqq u - (jk\alpha) < \epsilon.$$

Finally, take $n = jk$.

Example. Every point in the interval $[-1, 1]$ is a limit point for the sequence $\{\sin n\}$.

Proof. Let b belong to $[-1, 1]$. Choose c from $[0, 2\pi]$ such that $\sin c = b$. Then, given $\epsilon > 0$, choose $\delta > 0$ such that

$$|\sin x - b| < \epsilon \quad \text{for } |x - c| < \delta.$$

We now apply Theorem 2, taking $\alpha = 1/2\pi$ and $u = c/2\pi$; there exists a natural number n such that

$$0 < \frac{c}{2\pi} - \left(\frac{n}{2\pi}\right) < \frac{\delta}{2\pi}\cdot$$

Or, $0 < c - 2\pi(n/2\pi) < \delta$. It follows that

$$|\sin\{2\pi(n/2\pi)\} - b| < \epsilon.$$

But

$$2\pi\left(\frac{n}{2\pi}\right) = 2\pi\left\{\frac{n}{2\pi} - \left[\frac{n}{2\pi}\right]\right\} = n - 2k\pi.$$

Thus, we have $|\sin n - b| < \epsilon$.

EXERCISE 1. Determine the radius of convergence of $\sum(\sin n)x^n$.

EXERCISE 2. Prove: Given any real number u, there exists an increasing sequence of natural numbers, say $\{n_k\}$, such that $\{\tan n_k\} \to u$.

EXERCISE 3. Prove: If f is a piecewise continuous, periodic function having an irrational period, then $\{f(n)\}$ is dense in the range of f.

Remark. Theorem 2 states that the sequence $\{(n\alpha)\}$, α irrational, is dense in

[0, 1]. It is of further interest that this sequence is not "more dense" in some parts of the interval than in others. That is, the points of $\{(n\alpha)\}$ are distributed in [0, 1] in such a manner that the following result holds: Let I be any subinterval of [0, 1], $I_n = \{(\alpha), (2\alpha), \cdots, (n\alpha)\}$, and N_n be the number of elements in $I \cap I_n$. Then $\{N_n/n\} \to$ length of I. We say that $\{(m\alpha)\}$ is *uniformly distributed* in $[-1, 1]$. Two proofs of this deeper result can be found in [**1**]. On the other hand, due to the nonlinearity of the sine function, the sequence $\{\sin n\}$ is not uniformly distributed in $[-1, 1]$.

Reference

1. Ivan Niven, Irrational Numbers, Carus Monograph 11, The Mathematical Association of America, 1956.

ON THE AUTOMORPHISMS OF THE COMPLEX NUMBER FIELD

T. SOUNDARARAJAN, Madurai University, India

P. B. Yale remarks in [2] that he has not seen a proof of the fact that the complex number field has $2^{2^{\aleph_0}}$ automorphisms. We give below a proof of the same.

THEOREM. *The complex number field C has* $2^{2^{\aleph_0}}$ *automorphisms.*

Proof. It is well known that the complex number field has $c(=2^{\aleph_0})$ elements. If B is any transcendence base for C over the field Q of rationals, then B must also have cardinal c since the cardinal of C is the same as the cardinal of $Q(B)$ [**1**, p. 143]. We first show that the number of automorphisms of C is $\geqq 2^c$ by associating with each subset of B an automorphism. Let S be any subset of B. Consider the field $Q(B)$ over the field $Q(S)$. The set $\{-x \mid x \in B \sim S\}$ is also a set of algebraically independent elements for $Q(B)$ over $Q(S)$ and so the map $x \in B \sim S \to -x$ yields an automorphism of $Q(B)$ leaving S fixed. This automorphism of $Q(B)$ which leaves S and all x^2, $x \in B \sim S$ fixed can be extended to an automorphism of the field C. (See Theorem 7 of [2].) Let us denote this automorphism by S_σ. Thus for each subset $S \subset B$ we get an automorphism S_σ. Obviously if $S \neq S'$, $S_\sigma \neq S'_\sigma$, there are at least as many automorphisms as there are subsets of B. Hence the number of automorphisms is $\geqq 2^c$. To show the other inequality we note first that the set of all automorphisms is a subset of the set of all mappings of C into C, i.e., c^c. But this has cardinality c^c which is equal to 2^c. Thus our result follows.

Regarding his comment 1, we may note that Jacobson [**1**, p. 157] remarks that if B is any transcendence base of C over Q then any 1-1 surjective mapping of B can be extended to an automorphism of C.

References

1. Nathan Jacobson, Lectures in Abstract Algebra III, Theory of Fields and Galois Theory, Van Nostrand, Princeton, 1964.

2. P. B. Yale, Automorphisms of the complex numbers, this MAGAZINE, 39 (1966) 135–141.

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References

1. Nathan Jacobson, Lectures in Abstract Algebra III, Theory of Fields and Galois Theory, Van Nostrand, Princeton, 1964.
2. P. B. Yale, Automorphisms of the complex numbers, this MAGAZINE, 39 (1966) 135–141.

APPLICATION OF FOURIER SERIES TO SUMMATION OF SERIES, I

CLARENCE R. EDSTROM, Air Force Institute of Technology

Rainville [1] states the following procedure for showing that

$$\sum_{n=0}^{\infty} \frac{(-1)^n \operatorname{Sech}[((2n+1)\pi)/2]}{2n+1} = \frac{\pi}{8}. \tag{1}$$

First, the solution of the two-dimensional steady-state temperature problem for a square flat plate with one edge held at temperature unity and the other edges held at zero must be determined. Then by superposition of solutions and by evaluation of the temperature at the center of the plate, (1) is obtained.

The purpose of this note is to show that (1) can be obtained independently of the temperature problem by the use of Fourier series. Consider the Fourier sine series expansion of $f(x) = \exp(ax)$, $0 \leq x \leq \pi$, where a is fixed,

$$\exp(ax) = \frac{2}{\pi} \sum_{m=1}^{\infty} \frac{m[1-(-1)^m \exp(a\pi)] \sin mx}{m^2 + a^2}, \quad 0 < x < \pi. \tag{2}$$

The evaluation of (2) at $x = \pi/2$ yields

$$\exp\left(\frac{a\pi}{2}\right) = \frac{2[1+\exp(a\pi)]}{\pi} \sum_{m=0}^{\infty} \frac{(-1)^m(2m+1)}{(2m+1)^2 + a^2}. \tag{3}$$

Rewriting (3) we obtain

$$\operatorname{Sech}\left(\frac{a\pi}{2}\right) = \frac{4}{\pi} \sum_{m=0}^{\infty} \frac{(-1)^m(2m+1)}{(2m+1)^2 + a^2}. \tag{4}$$

Now replacing a in (4) by $2n+1$, multiplying both sides by $((-1)^n)/(2n+1)$, and summing on n from 0 to ∞, we have

$$\sum_{n=0}^{\infty} \frac{(-1)^n \operatorname{Sech}\left[\frac{(2n+1)\pi}{2}\right]}{2n+1} = \frac{4}{\pi} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^{m+n}(2m+1)}{(2n+1)[(2m+1)^2 + (2n+1)^2]}. \tag{5}$$

Next consider

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)[(2m+1)^2 + (2n+1)^2]}$$

where m is fixed. Since this series is absolutely convergent, we can write

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)[(2m+1)^2 + (2n+1)^2]} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(2m+1)^2} - \sum_{n=0}^{\infty} \frac{(-1)^n(2n+1)}{(2m+1)^2[(2m+1)^2 + (2n+1)^2]}. \tag{6}$$

Then multiplying both sides of (6) by $(-1)^m(2m+1)$ and summing on m from 0 to ∞, we obtain

$$\tag{7} \begin{aligned} &\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^{m+n}(2m+1)}{(2n+1)[(2m+1)^2+(2n+1)^2]} \\ &= \sum_{m=0}^{\infty}\frac{(-1)^m}{2m+1}\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1} - \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^{m+n}(2n+1)}{(2m+1)[(2m+1)^2+(2n+1)^2]} . \end{aligned}$$

Also since the series of (4) is uniformly convergent for all a, taking the limit as $a\to 0$ in (4) yields

$$\tag{8} \sum_{m=0}^{\infty}\frac{(-1)^m}{2m+1}=\frac{\pi}{4} .$$

Substituting the result of (8) into (7), we have

$$\tag{9} \begin{aligned} &\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^{m+n}(2m+1)}{(2n+1)[(2m+1)^2+(2n+1)^2]} \\ &\qquad + \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^{m+n}(2n+1)}{(2m+1)[(2m+1)^2+(2n+1)^2]} = \frac{\pi^2}{16} . \end{aligned}$$

Interchanging the dummies m and n in the second double sum, we obtain

$$\tag{10} \begin{aligned} &\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^{m+n}(2m+1)}{(2n+1)[(2m+1)^2+(2n+1)^2]} \\ &\qquad + \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^{m+n}(2m+1)}{(2n+1)[(2m+1)^2+(2n+1)^2]} = \frac{\pi^2}{16} . \end{aligned}$$

Knopp [2] states the following theorem concerning the interchangeability of the order of summation of a double sum.

THEOREM: *Given a convergent series* $\sum_{m=0}^{\infty} a_m$ *where* $a_m=\sum_{n=0}^{\infty} b_{mn}$. *If* $\sum_{m=0}^{\infty} b_{mn}$ *converges for every* n, *then* $\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} b_{mn}=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} b_{mn}$ *if and only if* $c_n\to 0$ as $n\to\infty$ *where* $c_n=\sum_{m=0}^{\infty}\sum_{k=n}^{\infty} b_{mk}$.

We will now show that this theorem allows us to interchange the order of summation of

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^{m+n}(2m+1)}{(2n+1)[(2m+1)^2+(2n+1)^2]} .$$

First since

$$\sum_{n=0}^{\infty}\frac{(-1)^n \operatorname{Sech}\left[\frac{(2n+1)\pi}{2}\right]}{(2n+1)}$$

converges, the double sum of (5) also converges. Thus from (10) we observe that we have a convergent series. Next, for n fixed

$$\sum_{m=0}^{\infty} \frac{(-1)^{m+n}(2m+1)}{(2n+1)[(2m+1)^2+(2n+1)^2]}$$

is a convergent series. For the double sum under consideration c_n takes the form

$$c_n = \sum_{m=0}^{\infty} \sum_{k=n}^{\infty} \frac{(-1)^{m+k}(2m+1)}{(2k+1)[(2m+1)^2+(2k+1)^2]} \tag{11}$$

or

$$\begin{aligned} c_n = {} & \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^{m+k}(2m+1)}{(2k+1)[(2m+1)^2+(2k+1)^2]} \\ & - \sum_{m=0}^{\infty} \sum_{k=0}^{n-1} \frac{(-1)^{m+k}(2m+1)}{(2k+1)[(2m+1)^2+(2k+1)^2]} . \end{aligned} \tag{12}$$

Since the second summation of (12) is uniformly convergent for all n, taking the limit as $n \to \infty$ in (12) yields $\lim_{n\to\infty} c_n = 0$. Hence, the theorem gives us

$$\begin{aligned} & \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(-1)^{m+n}(2m+1)}{(2n+1)[(2m+1)^2+(2n+1)^2]} \\ & \qquad = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^{m+n}(2m+1)}{(2n+1)[(2m+1)^2+(2n+1)^2]} . \end{aligned} \tag{13}$$

Utilizing the result of (13) with (10), we obtain

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^{m+n}(2m+1)}{(2n+1)[(2m+1)^2+(2n+1)^2]} = \frac{\pi^2}{32} . \tag{14}$$

Thus substituting (14) into (5), we have

$$\sum_{n=0}^{\infty} \frac{(-1)^n \operatorname{Sech}\left[\frac{(2n+1)\pi}{2}\right]}{2n+1} = \frac{\pi}{8} . \tag{15}$$

References

1. E. D. Rainville, Elementary Differential Equations, 3rd Ed. Macmillan, New York, 1964.
2. Konrad Knopp, Infinite Sequences and Series, Dover, New York, 1956.

Addendum to 'On Finite Rings' (this MAGAZINE, March–April 1967). The following theorem should appear after Corollary 5, page 84:

THEOREM 2. *If n is a square-free integer then there exists $\tau(n)$ nonisomorphic finite rings, where $|R| = n$. All such rings are commutative.*

Proof. If n is a square-free integer, then any group of order n is metacyclic [2, page 148], hence all its Sylow-subgroups are cyclic. In particular if the group is abelian, this implies that the group itself is cyclic. The above remarks show that for a finite ring satisfying the hypothesis of the theorem its additive group is cyclic and hence by Corollary 2 the theorem follows.

$$\sum_{m=0}^{\infty} \frac{(-1)^{m+n}(2m+1)}{(2n+1)[(2m+1)^2+(2n+1)^2]}$$

is a convergent series. For the double sum under consideration c_n takes the form

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or

$$c_n = \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^{m+k}(2m+1)}{(2k+1)[(2m+1)^2+(2k+1)^2]} - \sum_{m=0}^{\infty} \sum_{k=0}^{n-1} \frac{(-1)^{m+k}(2m+1)}{(2k+1)[(2m+1)^2+(2k+1)^2]}. \tag{12}$$

Since the second summation of (12) is uniformly convergent for all n, taking the limit as $n \to \infty$ in (12) yields $\lim_{n\to\infty} c_n = 0$. Hence, the theorem gives us

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(-1)^{m+n}(2m+1)}{(2n+1)[(2m+1)^2+(2n+1)^2]} = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^{m+n}(2m+1)}{(2n+1)[(2m+1)^2+(2n+1)^2]}. \tag{13}$$

Utilizing the result of (13) with (10), we obtain

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^{m+n}(2m+1)}{(2n+1)[(2m+1)^2+(2n+1)^2]} = \frac{\pi^2}{32}. \tag{14}$$

Thus substituting (14) into (5), we have

$$\sum_{n=0}^{\infty} \frac{(-1)^n \operatorname{Sech}\left[\frac{(2n+1)\pi}{2}\right]}{2n+1} = \frac{\pi}{8}. \tag{15}$$

References

1. E. D. Rainville, Elementary Differential Equations, 3rd Ed. Macmillan, New York, 1964.
2. Konrad Knopp, Infinite Sequences and Series, Dover, New York, 1956.

Addendum to 'On Finite Rings' (this MAGAZINE, March–April 1967). The following theorem should appear after Corollary 5, page 84:

THEOREM 2. *If n is a square-free integer then there exists $\tau(n)$ nonisomorphic finite rings, where $|R| = n$. All such rings are commutative.*

Proof. If n is a square-free integer, then any group of order n is metacyclic [2, page 148], hence all its Sylow-subgroups are cyclic. In particular if the group is abelian, this implies that the group itself is cyclic. The above remarks show that for a finite ring satisfying the hypothesis of the theorem its additive group is cyclic and hence by Corollary 2 the theorem follows.

BOOK REVIEWS

Edited by Dmitri Thoro, San Jose State College

Materials intended for review should be sent to: Dmitri Thoro, Department of Mathematics, San Jose State College, San Jose, California 95114.

Geometry: A Modern Introduction. By Mervin L. Keedy and Charles W. Nelson. Addison-Wesley, Reading, Massachusetts, 1965. ix+324 pp. $7.95.

The authors state in the preface that this book provides a treatment of the foundations of geometry at a lower level than is usual, one which is appropriate for "prospective elementary or secondary school teachers as well as others who may be majoring or minoring in mathematics." One must conclude from the topics included that the book will most often be used for prospective teachers.

The task which the authors have tackled is a difficult one. The problem is one of balancing off rigor with intuition so that the treatment is reasonably accurate and yet will serve as a textbook. For the most part they have succeeded.

The notation is generally good. For example, the notation which is used to distinguish between open segments, half-open segments, rays, and lines is very useful. Only occasionally do the terminology and notation tend to get out of hand. For example, the definition of a plane and the treatment of the interior of an angle will, no doubt, prove difficult for the student for which the book is intended.

Chapter 1 gives a short historical introduction and discussion of axiomatic systems. Chapters 2 and 3 provide a fairly detailed treatment of points, lines, planes, separations, curves, and surfaces. Chapter 4 covers congruence and assorted topics such as map projections and an introduction to conic sections. Chapter 6 contains material on parallelism and similarity and provides an introduction to trigonometry. Chapter 7 briefly introduces some of the usual noneuclidean geometries and Chapter 10 gives a short, uncomplicated treatment of coordinate geometry.

Chapters 5, 8, and 9 deal with systems of measures and would seem to belong to a book on geometry only if one considers the type of student for whom the book is written, namely, the prospective teacher. The material in Chapter 5 on the calculation of volumes and Cavalieri's Principle is well written.

The style indicates the authors are indeed experienced teachers. The background, the examples and the historical notes are excellent. In view of the fact that the authors point out consistently throughout the book that there are alternatives available at many points and that they have chosen to select some particular definition or axiom over another, it seems that a student might resent the admonition on page 22 to memorize definitions.

There are some regrettable omissions of details. The most interesting property of the second cylindrical projection mentioned on page 113 is that it is area preserving; this fact, however, is not mentioned though other properties are. The conic sections are discussed in Chapter 4 as intersections of cones and planes. They are discussed again in Chapter 10 but with the usual locus definitions and the flat statement is made that the two definitions are equivalent, at

least in the case of a parabola. With little effort the equivalence of these two could have been shown with the demonstrations of Quetelet and Dandelin.

An appendix containing background material on number systems, sets and logic along with a list of axioms, definitions and theorems is included at the end of the book.

The format is attractive, without the distracting gimmickry of multicolor printing or garish binding which has become common in recent texts. The careful attention to illustrations, binding, and typesetting are up to the high standard which one has come to expect from this publisher.

G. L. ALEXANDERSON, University of Santa Clara

Introduction to Modern Mathematics. By Nathan J. Fine. Rand McNally, Chicago, 1965. XV+509 pp. $8.50.

This book has been written for the undergraduate student in social science and liberal arts, with a preparation in high school mathematics. The author has treated many topics in undergraduate mathematics in a clear yet rigorous manner with a large and interesting selection of illustrative exercises. Although this is an elementary book, it requires a substantial amount of mathematical maturity, and is therefore more suitable as a text for prospective teachers with at least a minor in mathematics than for college freshmen.

The book presents an introductory treatment of logic and set theory as preparation for a most unusual presentation of axiom systems in chapter three. The comparison of physical and conceptual planes sets the stage for a clear and modern treatment of axiomatics including an excellent development of Boolean algebras. Whether a student is encountering the concept of isomorphism for the first time or as one who feels comfortable with this most important idea, he will gain much insight into the structure and power of modern mathematics through the careful development presented here.

Additional chapters treat the real numbers; linear algebra; analytic geometry; measure, area, and integration; limits, continuity, and differentiation; and probability.

The book is a scholarly approach to a logical treatment of elementary topics of college mathematics which should provide a challenging and stimulating introduction to modern mathematics.

ROBERT PRUITT, San Jose State College

Famous Problems of Mathematics. By H. Tietze. Graylock, New York, 1965. xvi+367 pp. $10.00.

It is a rare occasion when the publisher's blurb on the book cover will constitute a good objective report of the book. As this is one of those rare times, we quote from the cover:

"Professor Tietze's book is based on a series of lectures given to students of all faculties at the University of Munich. It is addressed to a wide audience and should be of interest to all who want to learn more about the development of mathematics from classical antiquity to modern times in the best way possible,

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through a discussion of some of the difficult problems which have challenged the minds of the greatest mathematicians and have led to the development of new fields of mathematics. The book consists of fourteen chapters and a Postscript, each followed by Notes which amplify and extend the material in the text. It contains many references to the literature (supplemented in this edition by an extensive bibliography of readily available works in English and a list of expository papers from the American Mathematical Monthly) for those who want to go more deeply into the topics discussed. In addition, the book contains more than 150 line cuts and 18 color plates and halftones which clarify the discussion."

Contents:

This book is highly recommended for the interested but uninformed layman, the bright student, and even the professional mathematician (who will most likely encounter some new material) and should be contained in all libraries—including secondary school ones.

Professor D. Crowe has pointed out to me the amusing fact that the small error in the colored cover diagram of the German edition is still present in this edition and that this error is pointed out in a footnote on page 72.

M. S. KLAMKIN, Ford Scientific Laboratory

Sphereland. By Dionys Burger (translated by Cornelie J. Rheinboldt). Crowell, New York, 1965. 205 pp. $4.95.

Sphereland is probably the best thing that could have happened to *Flatland.* For those familiar with Abbott's classic "Romance of Many Dimensions," this will be an unqualified endorsement of the originality and clarity, humor and fascination, and sheer delight this little volume may certainly afford the reader. The book should fascinate students, teachers, mathematicians, and intelligent amateurs alike. This broad appeal does not detract from the effectiveness of the treatment, rather it seems to focus on the universal appeal of mathematics.

Clifton Fadiman in a delightful essay, "Meditations of a Mathematical through a discussion of some of the difficult problems which have challenged the minds of the greatest mathematicians and have led to the development of new fields of mathematics. The book consists of fourteen chapters and a Postscript, *each followed by Notes which amplify and extend the material in the text. It* contains many references to the literature (supplemented in this edition by an extensive bibliography of readily available works in English and a list of expository papers from the American Mathematical Monthly) for those who want to go more deeply into the topics discussed. In addition, the book contains more than 150 line cuts and 18 color plates and halftones which clarify the discussion."

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Clifton Fadiman in a delightful essay, "Meditations of a Mathematical

Moron" in his collection *Any Number Can Play*, suggests, "Let's start with a few questions. Never mind the answers: interest starts with curiosity, not solutions." Burger, and before him Abbott, has seized on this very point and has documented a curiosity and its attempts to first explain away the question and then later to find answers that provide rational explanations for the curious.

The book begins with a resume of Abbott's *Flatland*, which treated the experiences of a square in two-dimensional Euclidean space. Dr. Burger resumes the story with the square's grandson, the hexagon, and his experiences in a two dimensional world now older and wiser. The passage of time however has not enabled the inhabitants to accept any more readily than before new and innovative explanations of the space in which they live. The existence of other worlds and the curved expanding nature of their plane present intriguing questions to the hexagon.

Brother Alfred in a recent article in the *Mathematics Teacher* ("A Mathematician's Progress," December 1966) remarks about mathematics exposition in general, and texts in particular: "The reader who scans such material has no idea of the hours and days of cogitation spent on even minor points, the misdirections that led to blind alleys; the sudden inspirations, and all the various thrills and disappointments that make mathematics a truly human experience." However, here the reader is able to share finding questions, experimenting, analyzing, debating, retracting, restating, testing, and reasoning to explain their two-dimensional world and submitting their theories to a three-dimensional visitor for verification. It is this process of discovery that makes *Sphereland* a "truly human experience."

P. J. BOYLE, James Lick High School, California

Visual Topology. By W. Lietzmann (Translated by M. Bruckheimer). American Elsevier, New York, 1965. xi+169 pp. $5.00.

In his *Elementary Mathematics from an Advanced Standpoint*, Felix Klein wrote about topology: "It would be highly desirable to have an easily readable account, accessible to the beginner, which by simple examples introduced and developed the general ideas of the abstract theory." In *Visual Topology*, we have the fulfillment of Klein's desire. Lietzmann's style is both relaxed and informal. He deals with such concrete, visual, and tangible objects as threads, wires, trees, knots, etc., dispensing with rigorous analytic derivation of facts in favor of visual and experimental methods. The book presupposes only high school mathematics (i.e., algebra and plane and solid geometry) and is easily accessible to the beginner.

Lietzmann's use of numerous examples such as "knot-tying," electrical switching circuits, chemical structural formulas, sewing, knitting, traffic interchange planning, etc., shows the extensive applications of topological methods which are possible. In addition to its didactic value, the book is recreational and entertaining. After instructing the reader in the arts of "knot-tying," playing "string-games" and entering and leaving mazes, the author proceeds to discuss the possible movements of a knight on a chessboard, the game of dominoes and the classical problem of "crossing all seven bridges of Königsberg exactly once."

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In Part 2 of the book, Lietzmann proves Euler's Theorem on Polyhedra and discusses some of its applications. He illustrates the concepts of "one-sidedness" and "nonorientability" by showing the reader how to construct the Möbius strip and the Klein bottle. He continues with a good treatment of the classical (unsolved) "four color problem" in the plane, pointing out that the corresponding result for the torus is known. The author concludes with an intuitive discussion of the projective plane and Riemann surfaces. The result is a delightfully readable introduction to the combinatorial methods of studying the topological properties of "line-structures" and surfaces.

BENJAMIN SIMS, San Jose State College

PROBLEMS AND SOLUTIONS

EDITED BY ROBERT E. HORTON, Los Angeles City College

Readers of this department are invited to submit for solution problems believed to be new that may arise in study, in research, or in extra-academic situations. Proposals should be accompanied by solutions, when available, and by any information that will assist the editor. Ordinarily, problems in well-known textbooks should not be submitted.

Solutions should be submitted on separate, signed sheets. Figures should be drawn in India ink and exactly the size desired for reproduction.

Send all communications for this department to Robert E. Horton, Los Angeles City College, 855 North Vermont Avenue, Los Angeles, California 90029.

To be considered for publication, solutions should be mailed before December 1, 1967.

PROBLEMS

663. *Proposed by Charles W. Trigg, San Diego, California.*

Find a Pythagorean triangle with sides of three digits each such that the nine digits involved are distinct.

664. *Proposed by Bruce B. Peterson, Middlebury College, Vermont.*

If $\{p_n/q_n\}$ is a sequence of distinct rationals which converges to x and $\{s_n\}$ is a bounded sequence, then the sequence $(p_n+s_n)/q_n$ converges to x.

665. *Proposed by Gregory Wulczyn, Bucknell University, Pennsylvania.*

Show that the volume of the hypersolid formed by the n coordinate planes and the tangent plane at any point of the hypersurface $x_1x_2x_3 \cdots x_n - a^n = 0$ is a constant.

666. *Proposed by Dennis P. Geller, University of Michigan.*

Consider a rectangle R with sides x and y, $x<y$. Suppose that we remove from R an x by x square to get a rectangle R'. If R' is similar to R, we know that R is the golden rectangle and $y/x=(1+\sqrt{5})/2$. We will suppose, however, that R' is not similar to R. Removing as before a square with side the smaller of the two sides of R', we have a rectangle R''. Under what conditions is R'' similar to R?

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667. *Proposed by Lew Kowarski, Morgan State College, Maryland.*

In how many different ways can one place on a shelf N encyclopedia volumes so that no volume is in its proper place?

668. *Proposed by Walter W. Horner, Pittsburgh, Pennsylvania.*

A cook is making pancakes on a circular griddle 26 units in diameter. She poured batter for three circular pancakes of unequal size which completely covered a diameter of the griddle but only half of its area. Find the diameters of the pancakes in integers.

SOLUTIONS

Late Solutions

Editor's note. A large number of solvers have been submitting solutions too late to be considered for publication. Therefore a deadline for solutions to be considered for publication will appear in the heading of the Problems and Solutions Department. Note the December 1, 1967, deadline for this issue.

José Asseo, Cleveland, Ohio: **632, 634**; *Gladwin E. Bartel, Whitworth College, Washington:* **631**; *Joseph Bohac, St. Louis, Missouri:* **631**; *Dermot A. Breault, Harvard Computing Center:* **628, 629, 631**; *Paul J. Campbell, University of Dayton:* **625**; *Brother Thomas Flynn, St. Mary's College, California:* **637, 640**; *Louise S. Grinstein, New York, New York:* **631, 632**; *J. Ray Hanna, University of Colorado:* **631**; *R. A. Jacobson, Houghton College, New York:* **635**; *Alan W. Johnson, Jr., Towson, Maryland:* **634**; *Lawrence Kreicher, Lakewood, Ohio:* **638**; *John D. Kunlo, U. S. Naval Space Surveillance System, Dahlgren, Virginia:* **631**; *G. E. Lewer, University of Sydney, Australia:* **631**; *Alexandru Lupas, Institutul de Calcul, Cluj, Rumania:* **614**; *Prasert Na Nagara, Kasetsart University, Bangkok, Thailand:* **628, 629, 631, 633, 634**; *C. Stanley Ogilvy, Hamilton College, New York:* **640**; *E. J. F. Primrose, University of Leicester, England:* **634**; *Stanley Rabinowitz, Far Rockaway, New York:* **635, 636, 638, 639, 640**; *Kenneth A. Ribet, Brown University:* **641**; *Richard Riggs, Jersey City State College:* **633**; *Sister M. Stephanie Sloyan, Georgian Court College, New Jersey:* **637; 639, 640**; *Daniel R. Stark, Cleveland State University, Ohio:* **622, 627, 632, 634, 638**; *E. P. Starke, Plainfield, New Jersey:* **633, 634, 636, 638**; *Kao-Hwa Sze, Louisiana State University:* **641**; *Charles W. Trigg, San Diego, California:* **635, 636, 637**; *Dimitrios Valthis, Halcis, Greece:* **637**; *Jerry Waxman, Brooklyn, New York:* **621, 622**; *Dale Woods, Missouri State Teachers College, Kirksville, Missouri:* **628, 631, 633**; *Gregory Wulczyn, Bucknell University:* **637**; *S. P. Singh, University of Windsor, Canada:* **632**; *A. F. Beardon, University of Kent, England:* **630**; *Lowell Van Tassel and James Seiler:* **638** *(jointly)*.

The Sleigh Ride

642. [January, 1967] *Proposed by Maxey Brooke, Sweeny, Texas.*

I left Hooten-Holler and traveled west to Muletrack at 10 miles per hour. Sometime later, I returned to Hooten-Holler. My respective departure times were 8:00 a.m. and 10:00 a.m. local time. My respective arrival times were 9:25 a.m. and 11:35 a.m. local time. What was my probable means of transportation?

Solution by Mrs. A. C. Garstang, Boulder, Colorado.

My probable means of transportation was by sleigh, perhaps pursued by a polar bear.

The data clearly indicate that the real time of the journey was $1\frac{1}{2}$ hours, and there was a change of 5 minutes of longitude in the journey. If θ is the latitude at which the journey was made, and the radius of the earth is 4,000 miles, the distance, d, traveled is,

$$d = 4{,}000 \cos \theta[5/(4 \times 57.3)]$$

Here the distance traveled is $1\frac{1}{2}\times 10$ mi., and one minute of time corresponds to $\frac{1}{4}$ degrees of longitude. Solving for θ we get $\theta\approx 80°$ north or south latitude.

Also solved by Donald Batman, MIT Lincoln Laboratory; George Stein, Swarthmore College; Charles W. Trigg, San Diego, California; and the proposer.

Four solvers missed the latitude.

The Unbiased Coin

643. [January, 1967]. *Proposed by Richard L. Eisenman, U. S. Air Force Academy.*

Prove that the probability of a match (HH or TT) is $\frac{1}{2}$ if and only if at least one of the coins is unbiased. Generalize to more than two coins.

Solution by W. O. J. Moser, McGill University.

THEOREM. *Let $n\geqq 2$ coins fall H with probabilities $\frac{1}{2}+a_1, \cdots, \frac{1}{2}+a_n$ respectively, $|a_i|\leqq\frac{1}{2}$. Then the probability that an even number fall H equals $\frac{1}{2}$ if and only if $a_1a_2\cdots a_n=0$ (i.e., at least one of the coins is unbiased).*

We will use the following easily established lemma (see for example, An Introduction to Combinatorial Analysis, J. Riordan, 1958, p. 9).

LEMMA. *The number of arrangements of n plus and minus signs along a straight line which contain an even (odd) number of plus signs is 2^{n-1}.*

Proof of Theorem. The probability that an even number fall H is

$$f(a_1, a_2, \cdots, a_n) = \sum (1/2 \pm a_1)(1/2 \pm a_2) \cdots (1/2 \pm a_n)$$

where the sum is taken over the 2^{n-1} arrangements of n plus and minus signs containing an even number of plus signs. This polynomial is symmetric in the a_i and hence has the form

$$C_0 + C_1 \sum a_i + C_2 \sum_{i<j} a_ia_j + C_3 \sum_{i<j<k} a_ia_ja_k + \cdots + C_na_1a_2\cdots a_n.$$

Clearly

$$C_0 = 2^{-n}2^{n-1} = 1/2, \qquad C_n = (-1)^n 2^{n-1}.$$

Furthermore $C_k=0$ for $k=1, 2, \cdots, n-1$. This can be easily seen as follows. C_k is the coefficient of the term in $a_1a_2\cdots a_k$, in (2). A term of (1) in which the number of minus signs among the first k factors is even (odd) contributes 2^{k-n}, (-2^{k-n}) to C_k, and by the lemma there are $2^{k-1}2^{n-k-1}$ $(2^{k-1}2^{n-k-1})$ such terms. Hence

$$C_k = 2^{k-n}2^{n-2} - 2^{k-n}2^{n-2} = 0.$$

Thus $f(a_1, \cdots, a_n)=1/2+(-1)^n2^{n-1}a_1a_2\cdots a_n$, so $f(a_1, a \cdots, a_n)=1/2$ if and only if $a_1a_2\cdots a_n=0$.

Also solved by Merrill Barnebey, Wisconsin State University at LaCrosse; G. E. Bartel, Whitworth College, Washington; Harry M. Gehman, SUNY at Buffalo; Michael Goldberg, Washington, *D.C.; Harry R. Henshaw, Victoria, B. C., Canada; David C. Hoaglin, Princeton, New Jersey; Douglas H. Johnson, University of Wisconsin; and Richard Riggs, Jersey City State College.*

Using methods similar to those above, Gehman stated the theorem:

If $n-1$ coins out of n coins are unbiased, the probability that all the coins will match is $\frac{1}{2}^{n-1}$ even if the remaining coin is biased.

The converse of this theorem is not true. Consider the counterexample of three coins with probabilities of H given as $\frac{3}{8}$, $\frac{3}{4}$, and $\frac{3}{4}$ respectively.

Area Equated to Perimeter

644. [January, 1967] *Proposed by Harlan L. Umansky, Union City, New Jersey.*

Find all the rectangles in which the area and the perimeter equal the same integer. Do the same for right triangles, equilateral triangles, and squares.

Solution by J. S. Vigder, Defense Research Board of Canada.

Let the dimensions of the rectangle be x, y and N the integer which is the area and perimeter.

Then $xy=2(x+y)=N$.

$$y = N/x, \qquad 2(x + N/x) = N$$
$$2x^2 - Nx + 2N = 0$$
$$x = \frac{N \pm \sqrt{(N^2 - 16N)}}{4}$$

The dimensions of all rectangles satisfying the conditions are $\frac{1}{4}(N+\sqrt{N^2-16N})$ and $\frac{1}{4}(N-\sqrt{N^2-16N})$ for integers $N \geqq 16$.

The only rectangles with rational sides satisfying the conditions are those where $N=16$, 18 and 25 with dimensions (4, 4), (6, 3) and (10, $2\frac{1}{2}$) respectively.

For a right triangle let the two sides be x and y

$$\tfrac{1}{2}xy = x + y + \sqrt{(x^2 + y^2)} = N$$
$$y = (2N)/x$$
$$x + (2N)/x + \sqrt{x^2 + (4N^2)/(x^2)} = N$$
$$\sqrt{(x^4 + 4N^2)} = Nx - x^2 - 2N$$
$$x^4 + 4N^2 = N^2x^2 + x^4 + 4N^2 - 2Nx^3 + 4Nx^2 - 4N^2x.$$
$$2x^2 - (N + 4)x + 4N = 0.$$
$$x = \frac{N + 4 \pm \sqrt{[(N + 4)^2 - 32N]}}{4} = \frac{N + 4 \pm \sqrt{[(N - 12)^2 - 128]}}{4}$$

The sides of all such right triangles are $\frac{1}{4}(N+4+\sqrt{(N-12)^2-128})$, $\frac{1}{4}(N+4-\sqrt{(N-12)^2-128})$ for integers $N \geqq 24$.

The only solutions with rational sides are given when $N=24$, 30 and 45 with sides (8, 6), (12, 5) and (20, $4\frac{1}{2}$) respectively.

There is no equilateral triangle satisfying the conditions of the problem, since the only equilateral triangle with perimeter and area equal has side $4\sqrt{3}$ with the area and perimeter $12\sqrt{3}$.

The only square having the area and perimeter equal is the square of side 4.

An interesting case is to find all isosceles triangles satisfying the condition. If we let the base of the triangle be $2x$ and the side y we have

$$x\sqrt{(y^2 - x^2)} = 2(x + y) = N$$

$$y = \tfrac{1}{2}N - x$$

$$x\sqrt{(1/4N^2 - Nx)} = N$$

$$4x^3 - Nx^2 + 4N = 0$$

Any integral value of $N \geqq 21$ will yield two triangles satisfying the conditions of the problem. The only such triangle with rational sides is that with base 12 and side $7\frac{1}{2}$. The only other such triangle which can be constructed with ruler and compass from a unit length is that with base $\frac{3}{4}(\sqrt{33}+1)$ and side $\frac{3}{8}(35-\sqrt{33})$. For both these triangles $N=27$.

Also solved by Leon Bankoff, Los Angeles, California; Merrill Barnebey, Wisconsin State University at LaCrosse; John Beidler, Scranton University, Pennsylvania; Richard J. Bonneau, Holy Cross College, Massachusetts; S. Brooke, Utica Free Academy, New York; Madonna S. Chernesky, Bowie, Maryland; Steven R. Conrad, Francis Lewis High School, Flushing, New York; David M. Crystal, Clarkson College of Technology, New York; J. A. Darragh, Long Beach, State College California; Huseyin Demir, Middle East Technical University, Ankara, Turkey; Fred L. Ermis, Jr., Wharton County Junior College, Texas; Herta T. Freitag, Hollins, Virginia; Reinaldo E. Giudici, University of Pittsburgh; Michael Goldberg, Washington, D.C.; Richard A. Jacobson, Houghton College, New York: Lew Kowarski, Morgan State College, Maryland; Maurice Nadler, Pace College, New York; Michael W. O'Donnell, University of Missouri; Richard Riggs, Jersey City State College; Stephen Spindler, Purdue University; Michael J. Sheridan, San Diego, California; Daniel R. Stark, Cleveland State University; E. P. Starke, Plainfield, New Jersey; P. D. Thomas, U. S. Naval Oceanographic Office, Suitland, Maryland; Charles W. Trigg, San Diego, California; John Waddington, Levack, Ontario, Canada; Louis R. Wirak, Lake-Sumter Junior College, Florida; and the proposer.

Paul Yearout, Brigham Young University, Utah, referred to an article in *The Mathematics Teacher*, Vol. 58, No. 4, April, 1965, pp. 303–307 entitled: "Conditions Governing Numerical Equality of Perimeter, Area, and Volume," by Leander W. Smith, of the University of Oklahoma.

A Parallelogram

645. [January, 1967] *Proposed by Esther Szekeres, University of Sydney, Australia.*

Given a convex quadrilateral, we drop from each vertex perpendiculars to the two sides not passing through it. Prove that if the sum of the lengths of these pairs of perpendiculars is the same for each vertex, the quadrilateral is a parallelogram.

Solution by Charles W. Trigg, San Diego, California.

Consider a quadrilateral $aMbNcPdQa$, where the small letters are sides and the capital letters are vertices, in order. Let the interior angles at Q and N be θ and ϕ, respectively. Then the perpendiculars from M and P are

$$(1) \qquad a \sin\theta + b \sin\phi = d \sin\theta + c \sin\phi$$

so

$$(2) \qquad (a - d) \sin\theta = (c - b) \sin\phi$$

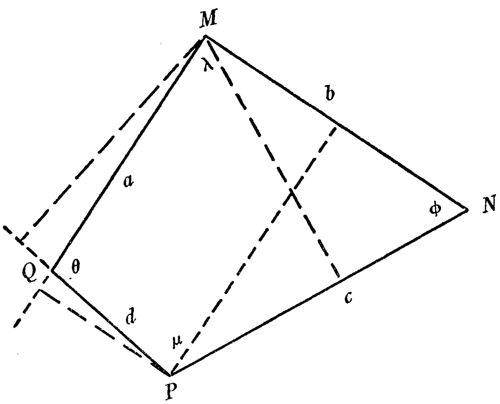
or

$$(3) \qquad (a - d)/(c - b) = \sin\phi/\sin\theta$$

Now consider the quadrilateral to be hinged at the vertices. The sides have fixed length, the angles change, so the left member of (3) is constant, the right member varies. Hence the only way that (2) can hold is if $(a-d)=0$ and $(c-b)=0$, or if $\sin\phi=\sin\theta$.

With angles λ and μ at M and P, a similar argument leads to $a=b$, $c=d$ or $\sin\lambda=\sin\mu$.

Thus either $a=b=c=d$ and the quadrilateral is a rhombus or the opposite angles of the quadrilateral are equal or supplementary. But if they were supplementary, the quadrilateral would be inscriptible, and the sums of the perpendiculars would be equal only if it were a rectangle. So, in any event, the quadrilateral must be a parallelogram.



Also solved by Leonard Carlitz, Duke University; Huseyin Demir, Middle East Technical University; Michael Goldberg, Washington, D. C.; Sidney H. L. Kung, Jacksonville University, Florida; Joseph Malkevitch, University of Wisconsin; and the proposer.

The Complete Quadrilateral

646. [January, 1967] *Proposed by V. F. Ivanoff, San Carlos, California.*

Denoting the pairs of opposite vertices of a complete quadrilateral by A and A', B and B', C and C', respectively, prove that

$$\frac{AB \cdot AB'}{A'B \cdot A'B'} = \frac{AC \cdot AC'}{A'C \cdot A'C'}.$$

I. *Solution by Huseyin Demir, Middle East Technical University, Ankara, Turkey.*

Writing the equality in the form

$$(AB/AC)(AB'/AC')(A'C'/A'B)(A'C/A'B') = 1$$

and replacing each fraction by its equivalent given by the sine law we have

$$(\sin C/\sin B)(\sin C'/\sin B')(\sin B/\sin C')(\sin B'/\sin C) = 1$$

which is an identity.

II. *Solution by Bruce W. King, Burnt Hills-Ballston Lake High School, New York.*

Line $A'C'$ meets sides AB, BC and CA of triangle ABC at the collinear points C', A' and B', respectively. By Menelaus' Theorem it follows that

(1) $$\frac{AC'}{C'B}\cdot\frac{BA'}{A'C}\cdot\frac{CB'}{B'A} = 1.$$

Similarly, line $A'B$ meets the sides of triangle $AB'C'$ at collinear points so that

(2) $$\frac{AB}{BC'}\cdot\frac{C'A'}{A'B'}\cdot\frac{B'C}{CA} = 1.$$

Dividing (2) into (1) yields an equation that is clearly equivalent to the desired result.

If, furthermore, A, B, and C are collinear, then by interchanging B for B' or C for C', the proof above will still be valid.

Also solved by Leon Bankoff, Los Angeles, California; Louise S. Grinstein, New York, New York; Maurice Nadler, Pace College, New York; Sister Stephanie Sloyan, Georgian Court College, New Jersey; Charles W. Trigg, San Diego, California; Gregory Wulczyn, Bucknell University; and the proposer.

Cubes of Digits

647. [January, 1967) *Proposed by C. R. J. Singleton, Petersham, Surrey, England.*

Consider any nonnegative number which is a multiple of 3. Calculate the sum of the cubes of its digits. Calculate the sum of the cubes of the digits of this new number. Repeat this process indefinitely. Prove that any initial number will eventually generate the number 153.

Solution by Robert W. Prielipp, University of Wisconsin.

For each positive integer n let $C(n)$ be the sum of the cubes of the decimal digits of n.

We establish initially that if n is divisible by 3 then $C(n)$ is divisible by 3. Let $n = a_1a_2 \cdots a_k$ where $a_1a_2 \cdots a_k$ is a base ten numeral (not a product). Since n is divisible by 3, $a_1+a_2+ \cdots +a_k \equiv 0 \pmod 3$. Thus $(a_1+a_2+ \cdots +a_k)^3 \equiv 0$

(mod 3), or $a_1^3+a_2^3+\cdots+a_k^3\equiv 0$ (mod 3) since in a field of characteristic $p (x+y)^p=x^p+y^p$. It is an immediate corollary that if n is divisible by 3 then $c^j(n)$ is divisible by 3. It is known (see Solution of Problem E 1810, *The American Mathematical Monthly*, 74 (January, 1967), 87–88, that for each positive integer n there exists a positive integer t such that $C^t(n)\in\{1, 55, 133, 136, 153, 160, 217, 244, 250, 352, 370, 371, 407, 919, 1459\}$. Therefore the desired result follows since 153 is the only element in this set which is a multiple of 3.

Also solved by Harry R. Henshaw, Victoria, B. C., Canada; Lew Kowarski, Morgan State College, Maryland; Lawrence A. Rossow, Gustavus Adolphus College, Minnesota; Daniel R. Start, Cleveland State College; E. P. Starke, Plainfield, New Jersey; Charles W. Trigg, San Diego, California; John W. Warren, Odessa, Texas; and the proposer.

Trigg found references in B. M. Stewart, "Sums of Functions of Digits," Canadian Journal of Mathematics, 12 (1960) 374–379, and Waclaw Sierpinski, Elementary Number Theory, Warszawa, 1964, p. 268.

A Property of Triangles

648. [January, 1967] *Proposed by Simeon Reich, Haifa, Israel.*

In the triangle $A_1A_2A_3$, let O_i be the midpoints of the sides and H_i the feet of the altitudes. Prove that

$$O_1H_2 + O_2H_3 + O_3H_1 = H_1O_2 + H_2O_3 + H_3O_1.$$

I. *Solution by Larry Hoehn, Perryville, Missouri.*

In triangle $A_1A_2A_3$, consider the right triangle $A_1H_3A_3$ (which may be formed by extending the sides of triangle $A_1A_2A_3$). Then O_2 is the midpoint of the hypotenuse A_1A_3; hence, $O_2H_3=1/2A_1A_3$ (since the midpoint of the hypotenuse is equidistant from the three vertices of a right triangle). In a similar manner, we have the following:

In the right triangle $A_3H_1A_1$, $O_2H_1 = \frac{1}{2}A_1A_3$,

In the right triangle $A_2H_3A_3$, $O_1H_3 = \frac{1}{2}A_2A_3$,

In the right triangle $A_3H_2A_2$, $O_1H_2 = \frac{1}{2}A_2A_3$,

In the right triangle $A_1H_2A_2$, $O_3H_2 = \frac{1}{2}A_1A_2$,

In the right triangle $A_2H_1A_1$, $O_3H_1 = \frac{1}{2}A_1A_2$.

Hence, $O_1H_2+O_2H_3+O_3H_1=O_2H_1+O_3H_2+O_1H_3$.

II. *Solution by Sister Stephanie Sloyan, Georgian Court College, New Jersey.*

With O_3 as center and O_3A_1 as radius, describe a circle. This circle will intersect side A_2A_3 in H_1 since an angle inscribed in a semicircle is a right angle. Similarly, the circle will intersect side A_1A_3 in H_2. Then $O_3H_1=H_2O_3$ because they are radii of the same circle. In like manner it can be shown that $O_2H_3=H_1O_2$ and $O_1H_2=H_3O_1$. The addition of these three equations gives the desired result.

Also solved by Leon Bankoff, Los Angeles, California; Lloyd F. Botway, Tufts University; Leonard Carlitz, Duke University; Huseyin Demir, Middle East Technical University, Ankara, Turkey; Michael Goldberg, Washington, D.C.; Louise S. Grinstein, New York, New York; Erwin

Just, Bronx Community College, New York; E. P. Starke, Plainfield, New Jersey; P. D. Thomas, U.S. Naval Oceanographic Office, Suitland, Maryland; Charles W. Trigg, San Diego, California; John Waddington, Levack, Ontario, Canada; Gregory Wulczyn, Bucknell University (two solutions); and the proposer.

Comment on Problem 622

622. [May, 1966, and January, 1967] *Proposed by Charles W. Trigg, San Diego, California.*

In various lands, there are discothèques with the name *Whisky A-Go-Go.* In the name of the Caribbean one,

$$RUM = AGO + GO,$$

each letter uniquely represents a digit in the scale of six. Rock "*n*" roll out the solution.

Comment by Harry M. Gehman, SUNY at Buffalo.

My only objection to the printed solution is that it leans too heavily on the fact that the base is six. It is just as easy to use any base.

If this problem is considered for a general base r, there are just two cases:

(I) $2O = M, \quad 2G = U + r, \quad A + 1 = R;$

and

(II) $2O = M + r, \quad 2G + 1 = U + r, \quad A + 1 = R.$

Trying various values of r, we find these solutions:

r	R	U	M	A	G	O	Case
6	5	0	2	4	3	1	I
7	4	5	2	3	6	1	I
7	6	0	1	5	3	4	II
7	2	0	5	1	3	6	II

For $r=8$, there are five solutions for Case I and eight solutions for Case II. Since the number of solutions tends to increase as r increases, there is no point in carrying the table any further.

Trigg's problem may also be revised as follows:

$$W\ S\ K\ Y = A\ G\ O + G\ O$$

The solution of this problem also leads to two cases:

I $\quad 2O = Y, \quad 2G = K + r, \quad A = r - 1, \quad S = 0, \quad W = 1;$

and

II $\quad 2O = Y + r, \quad 2G + 1 = K + r, \quad A = r - 1, \quad S = 0, \quad W = 1.$

Since there are seven letters, r must be at least 7. Trying various values of r, we find these solutions (note the unique solution for $r=7$):

r	W	S	K	Y	A	G	O	Case
7	1	0	3	4	6	5	2	I
8	1	0	2	6	7	5	3	I
8	1	0	3	4	7	5	6	II
9	1	0	5	4	8	7	2	I
9	1	0	3	4	8	6	2	I
9	1	0	5	6	8	7	3	I
9	1	0	2	3	8	5	6	II
9	1	0	4	5	8	6	7	II

For $r=10$, there are six solutions, three for each case. Again the number of solutions tends to increase with r so that there is no point in carrying this any further.

Comment on Problem 625

625. [May, 1966, and January, 1967] *Proposed by Roy Feinman, Rutgers University.*

Consider n independent events. Let their probabilities of occurring be $(\frac{1}{2})^n$, i.e., $1/2, 1/4, \cdots, 1/2^n$. What is the limiting value of the probability that at least one of them occurs, as $n \to \infty$?

Comment by Gerald V. McWilliams and Ray W. Thompson, LTV Electrosystems, Inc., Greenville, Texas.

In the solution of Problem 625, Messrs. James R. Kuttler and Nathan Rubenstein point out that

$$\prod_{k=1}^{\infty} (1 + q^{2^k}) = \frac{1}{1-q}$$

is often misprinted, in fact the corrected version given in the problem was misprinted, i.e.,

$$\prod_{k=1}^{\infty} (1 + q^{2k}) = \frac{1}{1-q}$$

As a further note, one can show by means of Maclaurin Series, that the solution to the problem may be expressed as

$$1 - \prod_{k=1}^{\infty} (1 - q^k) = \sum_{j=1}^{\infty} (-1)^{j-1} \frac{q^{j(j+1)/2}}{\prod_{i=1}^{j} (1 - q^i)}$$

As one can easily see, the series converges rather rapidly. As a quick bound, using, for $1 > a_j > 0$,

$$1 - \sum_{j=1}^{\infty} a_j \leqq \prod_{j=1}^{\infty} (1 - a_j) \leqq e^{-\sum_{j=1}^{\infty} a_j}$$

one has

$$.63212 \leqq 1 - e^{-1} \leqq 1 - \prod_{k=1}^{\infty} (1 - (1/2)^k) < 1.$$

A number of other solvers pointed out the misprint in the "corrected" formula.

Comment on Problem 629

629. [September, 1966, and March, 1967] *Proposed by C. Stanley Ogilvy and Stephen Barr, Hamilton College, New York.*

Rectangle $OPQR$ is initially placed so that OP lies along the positive x-axis and OR lies along the positive y-axis. If the rectangle is rotated through 90° in such a way that O slides along the x-axis and R slides along the y-axis, what is the locus of Q?

Comment by Leon Bankoff, Los Angeles, California.

This is a special case of the more general Locus Problem of Franciscus van Schooten (1615–1660): "Two vertices of a rigid triangle in a plane slide along the arms of an angle of the plane; what locus does the third vertex describe?" The locus is shown to be an ellipse. A discussion and solution of the generalized van Schooten version may be found in (1) Heinrich Dorrie, *100 Great Problems of Elementary Mathematics*, Dover, New York, 1965, pp. 214–216; (2) Edouard Callandreau, *Celebres Problemes Mathematiques*, Ed. Albin Michel, Paris, 1949, pp. 230–232.

Comment on Q 394

Q394. [November, 1966] Given a line segment AB of length $2k$. Find the area of the plane ring whose outer circle goes through A and B, and inner circle is tangent to AB. [*Submitted by Vladimir F. Ivanoff*]

Comment by Lyle E. Pursell, Grinnell College, Iowa.

In this problem the inner circle may be considered to be the locus of the midpoints of the set of all chords of the outer circle of constant length $2k$. Ivanoff's problem and solution has the following generalization:

> Let C be any piecewise smooth, closed, convex curve and let C_2 be the locus of midpoints of the set of all chords of the curve C of constant length $2\,k$. Then the area of the region between C and C_2 is $\pi\, k^2$.

That is, the area is independent of the curve C and depends only on the length of the generating chord. I do not know where this result may be found in the literature. I first learned of it from a lecture on "Some Remarkable Theorems About Areas" by L. R. Ford, given to the Indiana Section of the MAA at Purdue University on May 16, 1947. Professor Ford also considered a version in which the generating point was not the midpoint of the generating chord.

QUICKIES

From time to time this department will publish problems which may be solved by laborious methods, but which with the proper insight may be disposed of with dispatch. Readers are urged to submit their favorite problems of this type, together with the elegant solution and the source, if known.

Q412. Show that the inequality $|p(x)/q(x)| < a^2$, $(a \neq 0)$ can be solved without including the condition $q(x) > 0$ (or $q(x) < 0$) when determining the solution interval(s).

[*Submitted by Roger L. Opp*]

Q413. Prove or disprove: If $\lim_{x \to x_0} f(x)$ exists for each x in $[a, b]$, then $\int_a^b f(x)dx$ exists.

[*Submitted by Linda Pleska and J. F. Leetch*]

Q414. How many primes p exist such that p, $p+2d$ and $p+4d$ are all primes where d is not divisible by 3?

[*Submitted by Murray S. Klamkin*]

Q415. Show that if A is an n square matrix and each row (column) sums to c, then c is a characteristic root of A.

[*Submitted by Clarence C. Morrison*]

(Answers on page 199)

ANNOUNCEMENT OF L. R. FORD AWARDS

At its meeting on January 27, 1965, in Denver, Colorado, the Board of Governors authorized a number of awards, to be named after Lester R. Ford, Sr., to authors of expository articles published in the MONTHLY and the MATHEMATICS MAGAZINE. A maximum of six awards will be made annually; each award is in the amount of $100. The articles are to be selected by a subcommittee of the Committee on Publications appointed for this purpose.

The 1967 recipients of these awards, selected by a committee consisting of R. P. Boas, Chairman; C. W. Curtis, and R. P. Dilworth, were announced by President Moise at the Business Meeting of the Association on August 29, 1967, at the University of Toronto. The recipients of the Ford Awards for articles published in 1966 were the following:

Wai-Kai Chen, Boolean Matrices and Switching Nets, this MAGAZINE, 39(1966) 1–8.

D. R. Fulkerson, Flow Networks and Combinatorial Operations Research, MONTHLY, 73(1966) 115–138.

Mark Kac, Can One Hear the Shape of a Drum?, MONTHLY, 73 (1966) Part II (Slaught Paper No. 11), 1–23.

M. Z. Nashed, Some Remarks on Variations and Differentials, MONTHLY, 73(1966) Part II (Slaught Paper No. 11), 63–76.

P. B. Yale, Automorphisms of the Complex Numbers, this MAGAZINE, 39(1966) 135–141.

HENRY L. ALDER, *Secretary*

References

1. H. L. Alder, n and $n+1$ consecutive integers with equal sums of squares, Amer. Math. Monthly, 69 (1962) 282–285.
2. Brother U. Alfred, n and $n+1$ consecutive integers with equal sums of squares, this MAGAZINE 35 (1962) 155–164.
3. ———, Consecutive integers whose sum of squares is a perfect square, this MAGAZINE, 37 (1964) 19–32.
4. Stanton Philipp, Note on consecutive integers whose sum of squares is a perfect square, this MAGAZINE, 37 (1964), 218–220.
5. L. E. Dickson, History of the Theory of Numbers, vol. 2, Chelsea, New York, 1952. p. 354.

ANSWERS

A412. It has often been assumed that the solution leads to three conditions, namely (1) $q(x)<0$, (2) $-a^2q(x)>p(x)$, and (3) $p(x)>a^2q(x)$. However, conditions (2) and (3) imply (1) for $a^2q(x)<-p(x)$ and $a^2q(x)<p(x)$. Adding these inequalities gives $2a^2q(x)<0$ which is equivalent to (1). A similar argument holds when $q(x)>0$.

A413. The statement is true and follows from a result of E. W. Chittenden (Note on functions which approach a limit at every point of an interval, *American Mathematical Monthly*, 25 (1918) 249) which states that under the hypotheses of the statement, the set of discontinuities of f is countable.

A414. Now p must be of the form 3, $3m+1$, or $3m+2$, while d must be of the form $3n+1$ or $3n+2$. Going through the six possibilities, we find there is only one prime, $p=3$.

A415. If the argument of the characteristic function of A is set equal to c, adding the first $n-1$ rows (columns) to the nth row (column) yields a determinant whose nth row (column) is zero. Thus c is a characteristic root of A.

(Quickies on page 232)

ERRATA. January, 1967, Page 30. The solution A 400 should read: "The only solution is $y=0$ since $D^nx^{2n}D^n\equiv x^nD^{2n}x^n$. This follows from $D^mx^m=x^mD^m+a_1x^{m-1}D^{m-1}+\cdots+a_m$ by Leibniz' Theorem and $x^rD^r=xD(xD-1)\cdots(xD-r+1)\cdots$".

March, 1967, Page 108. Parts (a) and (b) were omitted from the statement of Problem 611. Page 109, Part (b), Line 2, at the end a denominator 2 is omitted. Page 109, bottom, the first column of the table should read: "x, w, l, W, L." Page 110, first line, the numeral "1" should be the letter "l".

M. S. Klamkin found Problem 626 [May, 1966, and January, 1967] as a theorem in R. A. Johnson, *Advanced Euclidean Geometry*.

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ANNOUNCEMENT OF L. R. FORD AWARDS

At its meeting on January 27, 1965, in Denver, Colorado, the Board of Governors authorized a number of awards, to be named after Lester R. Ford, Sr., to authors of expository articles published in the MONTHLY and the MATHEMATICS MAGAZINE. A maximum of six awards will be made annually; each award is in the amount of $100. The articles are to be selected by a subcommittee of the Committee on Publications appointed for this purpose.

The 1967 recipients of these awards, selected by a committee consisting of R. P. Boas, Chairman; C. W. Curtis, and R. P. Dilworth, were announced by President Moise at the Business Meeting of the Association on August 29, 1967, at the University of Toronto. The recipients of the Ford Awards for articles published in 1966 were the following:

Wai-Kai Chen, Boolean Matrices and Switching Nets, this MAGAZINE, 39(1966) 1–8.

D. R. Fulkerson, Flow Networks and Combinatorial Operations Research, MONTHLY, 73(1966) 115–138.

Mark Kac, Can One Hear the Shape of a Drum?, MONTHLY, 73 (1966) Part II (Slaught Paper No. 11), 1–23.

M. Z. Nashed, Some Remarks on Variations and Differentials, MONTHLY, 73(1966) Part II (Slaught Paper No. 11), 63–76.

P. B. Yale, Automorphisms of the Complex Numbers, this MAGAZINE, 39(1966) 135–141.

HENRY L. ALDER, *Secretary*

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